

Demi-linear Analysis II

—Demi-distributions

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Abstract. Using the equicontinuity results in [1], we establish a basic theory of demi-distributions which is a natural development of the usual distribution theory.

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The distributions of L. Schwartz, the generalized distributions of Beurling and the ultradistributions of Roumieu are continuous linear functionals defined on some suitable spaces of test functions.

In this paper, $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$ is a space of test functions and a function $f : E \rightarrow \mathbb{C}$ is called a demi-distribution if f is continuous and demi-linear [1, 2]. Thus, the family of demi-distributions includes all usual distributions and various nonlinear functionals.

As was stated in [1, 2], the family of demi-linear mappings is a natural and valuable extension of the family of linear operators. The propositions in this paper show that the family of demi-linear mappings can be used to develop the theory of distributions. For instance, in the case of the usual distributions the simplest equation $y' = 0$ has solutions $y = \text{constant}$ only. However, we will show that the equation $y' = 0$ has extremely many solutions which are nonlinear demi-linear functionals, and the equation $y' = f$ also has extremely many solutions which are demi-distributions. Moreover, we will show that the family of demi-distributions is

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closed with respect to extremely many of nonlinear transformations such as $|f(\cdot)|$, $\sin |f(\cdot)|$, $e^{|f(\cdot)|} - 1$, etc.

The main purpose of this paper is to establish the basic facts for the demi-linear functionals defined on the spaces of test functions, e.g., the plenty of nonlinear demi-linear functionals, differentiations, Fourier transforms and convolutions, etc.

The new equicontinuity results in [1] are fundamental to the theory of demi-distributions in this paper and so this theory is a natural continuation of functional analysis. Hence, our starting point is quite different from the hyperfunctions of M. Sato and the operational calculus of J. Mikusinski, etc.

1 Demi-distributions

For $a > 0$, $\mathcal{D}_a = \{\xi \in \mathbb{C}^{\mathbb{R}^n} : \xi \text{ is infinitely differentiable and } \xi(x) = 0 \text{ whenever } |x| = \sqrt{x_1^2 + \cdots + x_n^2} > a\}$ has the locally convex Fréchet topology which is given by the norm sequence $\{\|\xi\|_p = \sup_{|x| \leq a} \max_{|q| \leq p} |D^q \xi(x)|\}_{p=0}^\infty$. Let $\mathcal{D} = \bigcup_{m=1}^\infty \mathcal{D}_m$ be the strict inductive limit of $\{\mathcal{D}_m\}$.

Let $\mathcal{S} = \{\xi \in \mathbb{C}^{\mathbb{R}^n} : \xi \text{ is infinitely differentiable and rapidly decreasing}\}$. With the norm sequence $\{\|\xi\|_p = \sup_{|k|, |q| \leq p, x \in \mathbb{R}^n} |x^k D^q \xi(x)|\}_{p=0}^\infty$, \mathcal{S} is a locally convex Fréchet space, where $k = (k_1, k_2, \dots, k_n)$ is a multi-index and $x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$.

The space \mathcal{D} is an (LF) space and so \mathcal{D} is both barrelled and bornological [3, p.222]. Thus, \mathcal{D} is C -sequential [3, p.118]. There is an important fact which says that every sequentially continuous linear operator from a C -sequential locally convex space to a locally convex space must be continuous [3, p.118]. We now improve this fact as follows.

For topological vector spaces X and Y , $\mathcal{L}_{\gamma, U}(X, Y)$ and $\mathcal{K}_{\gamma, U}(X, Y)$ are the families of demi-linear mappings [1, 2].

Theorem 1.1 *Let X, Y be locally convex spaces, $U \in \mathcal{N}(X)$ and $\gamma_0(t) = t$, $\forall t \in \mathbb{C}$. If X is C -sequential and $f \in \mathcal{K}_{\gamma_0, U}(X, Y)$ is sequentially continuous, then f must be continuous.*

Proof. Let $V \in \mathcal{N}(Y)$. Pick balanced convex neighborhoods $U_0 \in \mathcal{N}(X)$ and $V_0 \in \mathcal{N}(Y)$ such that $U_0 \subset U$, $V_0 \subset V$.

Let $W = f^{-1}(V_0)$. For $u, w \in U_0 \cap W$ and scalars α, β with $|\alpha| + |\beta| \leq 1$, it follows from $f \in \mathcal{K}_{\gamma_0, U}(X, Y)$ that

$$f(\alpha u + \beta w) = f(\alpha u) + \beta f(w) = s_1 f(u) + s f(w) \in s_1 V_0 + s V_0,$$

where $|s_1| \leq |\gamma_0(\alpha)| = |\alpha|$, $|s| \leq |\gamma_0(\beta)| = |\beta|$. Then $|s_1| + |s| \leq |\alpha| + |\beta| \leq 1$ and so $f(\alpha u + \beta w) \in V_0$, $\alpha u + \beta w \in U_0 \cap W$. This shows that $U_0 \cap W$ is both balanced and convex.

Let $x_k \rightarrow 0$ in X . Then $x_k \in U_0$ eventually. Since f is sequentially continuous, $f(x_k) \rightarrow f(0) = 0$ and $f(x_k) \in V_0$ eventually, i.e., $x_k \in W$ eventually. Thus, $x_k \in U_0 \cap W$ eventually and so $U_0 \cap W$ is a sequential neighborhood of $0 \in X$. Since X is C -sequential, $U_0 \cap W \in \mathcal{N}(X)$ and $f(U_0 \cap W) \subset V_0 \subset V$. This shows that f is continuous at 0.

Suppose $(x_\alpha)_{\alpha \in I}$ is a net in X such that $x_\alpha \rightarrow x \in X$. Pick an $\alpha_0 \in I$ for which $x_\alpha - x \in U$, $\forall \alpha \geq \alpha_0$. For $\alpha \geq \alpha_0$,

$$f(x_\alpha) = f(x + x_\alpha - x) = f(x) + s_\alpha f(x_\alpha - x), \quad |s_\alpha| \leq |\gamma_0(1)| = 1.$$

But $f(x_\alpha - x) \rightarrow f(0) = 0$ and so $f(x_\alpha) \rightarrow f(x)$. \square

Theorem 1.2 Suppose that $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$, $\gamma \in C(0)$, $U \in \mathcal{N}(E)$ and Y is a topological vector space. If $f, f_\nu \in \mathcal{L}_{\gamma, U}(E, Y)$ are continuous ($\nu = 1, 2, 3, \dots$) and $f_\nu(\xi) \rightarrow f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ uniformly for $\xi \in B$.

Proof. Since both \mathcal{D}_a and \mathcal{S} are Fréchet spaces having the Montel property, we only need to consider \mathcal{D} . Suppose that $B \subset \mathcal{D}$ is bounded but $\lim_\nu f_\nu(\xi) = f(\xi)$ is not uniform for $\xi \in B$. Then there exist $V \in \mathcal{N}(Y)$, $\{\xi_k\} \subset B$ and integers $\nu_1 < \nu_2 < \dots$ such that

$$f_{\nu_k}(\xi_k) - f(\xi_k) \notin V, \quad k = 1, 2, 3, \dots$$

Pick a balanced $W \in \mathcal{N}(Y)$ for which $W + W + W \subset V$. Since B is bounded in \mathcal{D} , $B \subset \mathcal{D}_m$ for some $m \in \mathbb{N}$ [3, p.219] and B is relatively compact in the Fréchet space \mathcal{D}_m [4, Th. 1.6.2]. By passing to a subsequence if necessary, we say that $\xi_k \rightarrow \xi \in \mathcal{D}_m$. Since $f_\nu(\eta) \rightarrow f(\eta)$ at each $\eta \in \mathcal{D}_m$, $\{f_\nu\}_1^\infty$ is pointwise bounded on \mathcal{D}_m and so $\{f_\nu\}_1^\infty$ is equicontinuous on \mathcal{D}_m by Th. 3.1 of [1]. By Cor. 3.1 of [1], $\lim_k f_\nu(\xi_k) = f_\nu(\xi)$ uniformly for $\nu \in \mathbb{N}$ and so there is a $k_0 \in \mathbb{N}$ such that $f_\nu(\xi_k) - f_\nu(\xi) \in W$ for all $\nu \in \mathbb{N}$ and $k > k_0$. Since $f : \mathcal{D} \rightarrow Y$ is continuous and $f_\nu(\xi) \rightarrow f(\xi)$, there exist $\nu_0, k_1 \in \mathbb{N}$ such that $f(\xi) - f(\xi_k) \in W$ for all $k > k_1$ and $f_\nu(\xi) - f(\xi) \in W$ for all $\nu > \nu_0$.

Pick an integer $k_2 \geq k_0 + k_1$ for which $\nu_k > \nu_0$ whenever $k > k_2$. Then for every $k > k_2$ we have that

$$\begin{aligned} f_{\nu_k}(\xi_k) - f(\xi_k) &= f_{\nu_k}(\xi_k) - f_{\nu_k}(\xi) + f_{\nu_k}(\xi) - f(\xi) + f(\xi) - f(\xi_k) \\ &\in W + W + W \subset V. \end{aligned}$$

This is a contradiction and so $\lim_\nu f_\nu(\xi) = f(\xi)$ uniformly for $\xi \in B$. \square

In general, $\mathcal{L}_{\gamma, U}(X, Y) \subsetneq \mathcal{W}_{\gamma, U}(X, Y)$ where Y is locally convex. Using Th. 4.1 of [1] instead of Th. 3.1 of [1], the above proof gives an improved result as follows.

Theorem 1.3 Let $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$, $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$. Let Y be a locally convex space and $f, f_\nu \in \mathcal{W}_{\gamma, U}(E, Y)$ are continuous, $\nu = 1, 2, 3, \dots$. If $f_\nu(\xi) \rightarrow f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ uniformly for $\xi \in B$.

For $\mathcal{K}_{\gamma_0, U}(X, Y)$, we have a more strong result as follows.

Theorem 1.4 Let $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$, $U \in \mathcal{N}(E)$ and Y be a locally convex space. Let $f_\nu \in \mathcal{K}_{\gamma_0, U}(E, Y)$ be continuous, $\forall \nu \in \mathbb{N}$. If $\lim_\nu f_\nu(\xi) = f(\xi)$ exists at each $\xi \in E$, then f is also a continuous mapping in $\mathcal{K}_{\gamma_0, U}(E, Y)$ and for every bounded $B \subset E$, $\lim_\nu f_\nu(\xi) = f(\xi)$ uniformly for $\xi \in B$.

Proof. Only need to consider $E = \mathcal{D}$. Let $\xi \in E$, $\eta \in U$ and $|t| \leq 1$. Then $f(\xi + t\eta) = \lim_\nu f_\nu(\xi + t\eta) = \lim_\nu (f_\nu(\xi) + s_\nu f_\nu(\eta))$, where $|s_\nu| \leq |\gamma_0(t)| = |t| \leq 1$. Say that $s_{\nu_k} \rightarrow s$. Then $|s| = \lim_k |s_{\nu_k}| \leq |\gamma_0(t)|$ and

$$f(\xi + t\eta) = \lim_k f_{\nu_k}(\xi + t\eta) = \lim_k (f_{\nu_k}(\xi) + s_{\nu_k} f_{\nu_k}(\eta)) = f(\xi) + s f(\eta).$$

Thus, $f \in \mathcal{K}_{\gamma_0, U}(E, Y)$.

Let $\xi_k \rightarrow \xi$ in \mathcal{D} . Then $\xi_k \rightarrow \xi$ in \mathcal{D}_m for some $m \in \mathbb{N}$ [3, p.219]. Since $f_\nu(\cdot) \rightarrow f(\cdot)$, $\{f_\nu\}_1^\infty$ is pointwise bounded on \mathcal{D}_m and, by Cor. 3.1 of [1], $\lim_k f_\nu(\xi_k) =$

$f_\nu(\xi)$ uniformly for $\nu \in \mathbb{N}$. Then $\lim_k f(\xi_k) = \lim_k \lim_\nu f_\nu(\xi_k) = \lim_\nu \lim_k f_\nu(\xi_k) = \lim_\nu f_\nu(\xi) = f(\xi)$. Thus, $f : \mathcal{D} \rightarrow Y$ is sequentially continuous and so f is continuous by Th. 1.1.

Now the desired follows from Th. 1.2. \square

Henceforth, $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$.

Definition 1.1 $f : E \rightarrow \mathbb{C}$ is called a *demi-distribution* if f is continuous and $f \in \mathcal{L}_{\gamma, U}(E, \mathbb{C})$ for some $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$.

Let $E^{(\gamma, U)}$ be the family of demi-distributions related to $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$. Let $[E^{(\gamma, U)}]$ be the span $(E^{(\gamma, U)})$ in \mathbb{C}^E , i.e., $[E^{(\gamma, U)}] = \{\text{finite sum } \sum t_k f_k : t_k \in \mathbb{C}, f_k \in E^{(\gamma, U)}\}$.

Let E' be the space of usual distributions, i.e., E' is the space of continuous linear functionals. Obviously, $E' \subset E^{(\gamma, U)}$, $\forall U \in \mathcal{N}(E)$, $\gamma \in C(0)$.

Example 1.1 (1) For every $f \in L^1_{loc}(\mathbb{R}^n)$ define $[f] : \mathcal{D} \rightarrow \mathbb{R}$ by $[f](\xi) = \int_{\mathbb{R}^n} |f(x)\xi(x)| dx$, $\xi \in \mathcal{D}$.

Let $\gamma \in C(0)$ and $\xi, \eta \in \mathcal{D}$, $|t| \leq 1$. For every $x \in \mathbb{R}^n$ there exists $\alpha(x) \in [-|t|, |t|]$ such that $|\xi(x) + t\eta(x)| = |\xi(x)| + \alpha(x)|\eta(x)|$ and

$$\begin{aligned} [f](\xi + t\eta) &= \int_{\mathbb{R}^n} |f(x)(\xi + t\eta)(x)| dx \\ &= \int_{\mathbb{R}^n} |f(x)| |\xi(x) + t\eta(x)| dx \\ &= \int_{\mathbb{R}^n} |f(x)| [|\xi(x)| + \alpha(x)|\eta(x)|] dx \\ &= \int_{\mathbb{R}^n} |f(x)\xi(x)| dx + \int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx \\ &= [f](\xi) + \int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx. \end{aligned}$$

If $\int_{\mathbb{R}^n} |f(x)\eta(x)| dx = 0$, then $0 \leq \int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx \leq \int_{\mathbb{R}^n} |\alpha(x)f(x)\eta(x)| dx \leq |t| \int_{\mathbb{R}^n} |f(x)\eta(x)| dx = 0$ and so $\int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx = 0 = 0[f](\eta)$, where $0 \leq |\gamma(t)|$.

If $\int_{\mathbb{R}^n} |f(x)\eta(x)| dx \neq 0$, then $|\frac{\int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx}{\int_{\mathbb{R}^n} |f(x)\eta(x)| dx}| \leq |t| \leq |\gamma(t)|$ and so $\int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx = s \int_{\mathbb{R}^n} |f(x)\eta(x)| dx = s[f](\eta)$ where $s = \frac{\int_{\mathbb{R}^n} \alpha(x) |f(x)\eta(x)| dx}{\int_{\mathbb{R}^n} |f(x)\eta(x)| dx}$, $|s| \leq |t| \leq |\gamma(t)|$. Thus,

$$[f](\xi + t\eta) = [f](\xi) + s[f](\eta), \quad |s| \leq |\gamma(t)|,$$

i.e., $[f] \in \mathcal{X}_{\gamma, \mathcal{D}}(\mathcal{D}, \mathbb{R}) \cap \mathcal{D}^{(\gamma, \mathcal{D})}$ but $[f]$ is not a usual distribution.

(2) Let $\mathcal{D}_1(\mathbb{R}) = \{\xi \in \mathbb{R}^{\mathbb{R}} : \xi \text{ is infinitely differentiable and } \xi(x) = 0 \text{ for } |x| > 1\}$. Let $\gamma(t) = \frac{\pi}{2}t$ for $t \in \mathbb{R}$ and $U = \{\xi \in \mathcal{D}_1(\mathbb{R}) : \max_{|x| \leq 1} |\xi(x)| < 1\}$. Define $f : \mathcal{D}_1(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$f(\xi) = \int_{-\infty}^{\infty} |\sin \xi(x)| dx, \quad \xi \in \mathcal{D}_1(\mathbb{R}).$$

It is easy to show that if $a \in \mathbb{R}$ and $u, t \in [-1, 1]$ then $\sin(a+tu) = \sin a + s \sin u$ with $|s| \leq \frac{\pi}{2}|t|$. Hence, for $\xi \in \mathcal{D}_1(\mathbb{R})$, $\eta \in U$ and $|t| \leq 1$ we have that

$$\begin{aligned}
f(\xi + t\eta) &= \int_{-\infty}^{\infty} |\sin[\xi(x) + t\eta(x)]| dx \\
&= \int_{-\infty}^{\infty} |\sin \xi(x) + \alpha(x) \sin \eta(x)| dx \quad (|\alpha(x)| \leq \frac{\pi}{2}|t|) \\
&= \int_{-\infty}^{\infty} [|\sin \xi(x)| + \beta(x) |\sin \eta(x)|] dx \quad (|\beta(x)| \leq |\alpha(x)| \leq \frac{\pi}{2}|t|) \\
&= \int_{-\infty}^{\infty} |\sin \xi(x)| dx + \int_{-\infty}^{\infty} \beta(x) |\sin \eta(x)| dx \\
&= \int_{-\infty}^{\infty} |\sin \xi(x)| dx + s \int_{-\infty}^{\infty} |\sin \eta(x)| dx \quad (|s| \leq \frac{\pi}{2}|t| = |\gamma(t)|) \\
&= f(\xi) + sf(\eta), \quad |s| \leq \frac{\pi}{2}|t| = |\gamma(t)|.
\end{aligned}$$

Thus, $f \in \mathcal{K}_{\gamma, U}(\mathcal{D}_1(\mathbb{R}), \mathbb{R}) \cap (\mathcal{D}_1(\mathbb{R}))^{(\gamma, U)}$ but f is not a usual distribution.

(3) For the case of $\mathbb{R}^n = \mathbb{R}$, we write that $\mathcal{S} = \mathcal{S}(\mathbb{R})$. Let $U = \{\eta \in \mathcal{S}(\mathbb{R}) : \sup_{x \in \mathbb{R}} |\eta(x)| < 1\}$ and $\gamma(t) = et$ for $t \in \mathbb{C}$. Then define $g : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$g(\xi) = \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx, \quad \xi \in \mathcal{S}(\mathbb{R}).$$

For $\xi \in \mathcal{S}(\mathbb{R})$, $\eta \in U$ and $|t| \leq 1$,

$$\begin{aligned}
g(\xi + t\eta) &= \sqrt{-1} \int_{-1}^1 (e^{|\xi(x) + t\eta(x)|} - 1) dx \\
&= \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)| + \alpha(x)|\eta(x)|} - 1) dx \quad (\alpha(x) \in [-|t|, |t|]) \\
&= \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)| + \alpha(x)|\eta(x)|} - e^{\alpha(x)|\eta(x)|} + e^{\alpha(x)|\eta(x)|} - 1) dx \\
&= \sqrt{-1} \int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx + \sqrt{-1} \int_{-1}^1 (e^{\alpha(x)|\eta(x)|} - 1) dx.
\end{aligned}$$

If $\int_{-1}^1 (e^{|\xi(x)|} - 1) dx = 0$, then $0 \leq \int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx \leq e^{|t|} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx = 0$ and so $\sqrt{-1} \int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx = 0 = \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx = g(\xi) = rg(\xi)$ where $r = 1$, $|r - 1| = 0 \leq |\gamma(t)|$. If $\int_{-1}^1 (e^{|\xi(x)|} - 1) dx \neq 0$, then $\sqrt{-1} \int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx = \frac{\int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx}{\int_{-1}^1 (e^{|\xi(x)|} - 1) dx} \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx$,

where

$$\begin{aligned}
& \left| \frac{\int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx}{\int_{-1}^1 (e^{|\xi(x)|} - 1) dx} - 1 \right| = \frac{|\int_{-1}^1 (e^{\alpha(x)|\eta(x)|} - 1)(e^{|\xi(x)|} - 1) dx|}{\int_{-1}^1 (e^{|\xi(x)|} - 1) dx} \\
&= \frac{|\int_{-1}^1 e^{\theta(x)\alpha(x)|\eta(x)|} \alpha(x) |\eta(x)| (e^{|\xi(x)|} - 1) dx|}{\int_{-1}^1 (e^{|\xi(x)|} - 1) dx} \quad (0 \leq \theta(x) \leq 1) \\
&\leq \frac{\int_{-1}^1 e^{|t|} |t| (e^{|\xi(x)|} - 1) dx}{\int_{-1}^1 (e^{|\xi(x)|} - 1) dx} \quad (\because |\alpha(x)| \leq |t|, |\eta(x)| \leq 1) \\
&= e^{|t|} |t| \leq e|t| = |\gamma(t)|. \quad (\because |t| \leq 1)
\end{aligned}$$

Thus, $\sqrt{-1} \int_{-1}^1 e^{\alpha(x)|\eta(x)|} (e^{|\xi(x)|} - 1) dx = r\sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx = rg(\xi)$ where $|r - 1| \leq |\gamma(t)|$.

If $g(\eta) = \sqrt{-1} \int_{-1}^1 (e^{|\eta(x)|} - 1) dx = 0$, then $\eta(x) = 0$ a.e. in $[-1, 1]$ and $g(\xi + t\eta) = \sqrt{-1} \int_{-1}^1 (e^{|\xi(x) + t\eta(x)|} - 1) dx = \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx = g(\xi) = rg(\xi) + sg(\eta)$ where $r = 1$ and $s = 0$, $|r - 1| = 0 \leq |\gamma(t)|$, $|s| = 0 \leq |\gamma(t)|$.

Suppose that $g(\eta) = \sqrt{-1} \int_{-1}^1 (e^{|\eta(x)|} - 1) dx \neq 0$. Then

$$\sqrt{-1} \int_{-1}^1 (e^{\alpha(x)|\eta(x)|} - 1) dx = \frac{\int_{-1}^1 (e^{\alpha(x)|\eta(x)|} - 1) dx}{\int_{-1}^1 (e^{|\eta(x)|} - 1) dx} g(\eta),$$

where

$$\begin{aligned}
\left| \frac{\int_{-1}^1 (e^{\alpha(x)|\eta(x)|} - 1) dx}{\int_{-1}^1 (e^{|\eta(x)|} - 1) dx} \right| &= \frac{|\int_{-1}^1 e^{\delta(x)\alpha(x)|\eta(x)|} \alpha(x) |\eta(x)| dx|}{\int_{-1}^1 e^{\theta(x)|\eta(x)|} |\eta(x)| dx} \quad (0 \leq \delta(x), \theta(x) \leq 1) \\
&\leq \frac{|\int_{-1}^1 e^{\delta(x)\alpha(x)|\eta(x)|} |\alpha(x)| |\eta(x)| dx|}{\int_{-1}^1 |\eta(x)| dx} \\
&\leq \frac{\int_{-1}^1 e^{|t|} |t| |\eta(x)| dx}{\int_{-1}^1 |\eta(x)| dx} \\
&= e^{|t|} |t| \leq e|t| = |\gamma(t)|.
\end{aligned}$$

Then $g(\xi + t\eta) = rg(\xi) + sg(\eta)$ where $|r - 1| \leq |\gamma(t)|$, $|s| \leq |\gamma(t)|$, i.e., $g \in \mathcal{L}_{\gamma, U}(\mathcal{S}(\mathbb{R}), \mathbb{C})$. Since $\xi_k \rightarrow \xi$ in \mathcal{S} implies that $\|\xi_k - \xi\|_0 = \sup_{x \in \mathbb{R}} |\xi_k(x) - \xi(x)| \rightarrow 0$ and so $g(\xi_k) = \sqrt{-1} \int_{-1}^1 (e^{|\xi_k(x)|} - 1) dx \rightarrow \sqrt{-1} \int_{-1}^1 (e^{|\xi(x)|} - 1) dx = g(\xi)$, i.e., $g : \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$ is continuous. Thus, $g \in (\mathcal{S}(\mathbb{R}))^{(\gamma, U)}$.

For every $C \geq 1$ and $\varepsilon > 0$, $\mathcal{K}_{C, \varepsilon}(\mathbb{R}, \mathbb{R})$ includes a lot of nonlinear functions. Pick a $h \in \mathcal{K}_{C, \varepsilon}(\mathbb{R}, \mathbb{R})$ and let $f(x + iy) = ih(|x + iy|)$, $\forall x + iy \in \mathbb{C}$. If $|u + iv| < \varepsilon$ and $|t| \leq 1$, then $f[x + iy + t(u + iv)] = ih(|x + iy + t(u + iv)|) = ih(|x + iy| + \alpha|u + iv|) = ih(|x + iy|) + sih(|u + iv|) = f(x + iy) + sf(u + iv)$, where $\alpha \in [-|t|, |t|] \subset [-1, 1]$ and $|s| \leq C|\alpha| \leq C|t|$. This shows that $f \in \mathcal{K}_{C, \varepsilon}(\mathbb{C}, \mathbb{C})$ and, therefore, $\mathcal{K}_{C, \varepsilon}(\mathbb{C}, \mathbb{C})$ also includes a lot of nonlinear functions.

Let $E^{[\gamma, U]} = \{f \in \mathcal{K}_{\gamma, U}(E, \mathbb{C}) : f \text{ is continuous}\}$. Then $E^{[\gamma, U]} \subset E^{(\gamma, U)}$.

Theorem 1.5 *If $A \subset E'$ is an equicontinuous family of distributions and $\varepsilon > 0$, then there is a $U \in \mathcal{N}(E)$ such that*

$$\{h \circ f : h \in \mathcal{L}_{\gamma, \varepsilon}(\mathbb{C}, \mathbb{C}), f \in A\} \subset E^{(\gamma, U)}, \quad \forall \gamma \in C(0),$$

$$\{h \circ f : h \in \mathcal{H}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C}), f \in A\} \subset E^{[\gamma, U]}, \quad \forall \gamma \in C(0).$$

Proof. Since A is equicontinuous, there is a $U \in \mathcal{N}(E)$ such that $|f(\eta)| < \varepsilon$, $\forall f \in A, \eta \in U$. Let $\xi \in E$, $\eta \in U$ and $|t| \leq 1$. For $h \in \mathcal{L}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C})$ and $f \in A$,

$$\begin{aligned} (h \circ f)(\xi + t\eta) &= h(f(\xi) + tf(\eta)) \\ &= r(h \circ f)(\xi) + s(h \circ f)(\eta), \quad |r - 1| \leq |\gamma(t)|, \quad |s| \leq |\gamma(t)|. \end{aligned}$$

Thus, $h \circ f \in \mathcal{L}_{\gamma, U}(E, \mathbb{C})$.

Suppose that $h \in \mathcal{L}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C})$ and $w_k \rightarrow w$ in \mathbb{C} . Then

$$\begin{aligned} \lim_k h(w_k) &= \lim_k h(w + w_k - w) = \lim_k h(w + \frac{2(w_k - w)}{\varepsilon} \frac{\varepsilon}{2}) \\ &= \lim_k [r_k h(w) + s_k h(\frac{\varepsilon}{2})], \end{aligned}$$

where $|r_k - 1| \leq |\gamma(\frac{2(w_k - w)}{\varepsilon})| \rightarrow 0$ and $|s_k| \leq |\gamma(\frac{2(w_k - w)}{\varepsilon})| \rightarrow 0$, i.e., $r_k \rightarrow 1$, $s_k \rightarrow 0$. Thus, $h(w_k) \rightarrow h(w)$, h is continuous. But $A \subset E'$ and so $h \circ f : E \rightarrow \mathbb{C}$ is continuous for $h \in \mathcal{L}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C})$ and $f \in A$. \square

Corollary 1.1 *If $f \in E'$ is a usual distribution and $\varepsilon > 0$, then there is a $U \in \mathcal{N}(E)$ such that*

$$\{h \circ f : h \in \mathcal{L}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C})\} \subset E^{(\gamma, U)}, \quad \forall \gamma \in C(0),$$

$$\{h \circ f : h \in \mathcal{H}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C})\} \subset E^{[\gamma, U]}, \quad \forall \gamma \in C(0).$$

Corollary 1.2 *If $A \subset E'$ is a pointwise bounded family of usual distributions, then for every $\varepsilon > 0$ there is a $U \in \mathcal{N}(E)$ such that*

$$\{h \circ f : h \in \mathcal{L}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C}), f \in A\} \subset E^{(\gamma, U)}, \quad \forall \gamma \in C(0),$$

$$\{h \circ f : h \in \mathcal{H}_{\gamma,\varepsilon}(\mathbb{C}, \mathbb{C}), f \in A\} \subset E^{[\gamma, U]}, \quad \forall \gamma \in C(0).$$

Proof. Both \mathcal{D}_a and \mathcal{S} are Fréchet spaces. So for the case of $E \in \{\mathcal{D}_a, \mathcal{S}\}$, A is equicontinuous by Th. 3.1 of [1].

Since \mathcal{D} is an (LF) space, \mathcal{D} is barrelled [3, p.222]. Then A is also equicontinuous for the case of $E = \mathcal{D}$ [3, Th. 9.3.4]. \square

Corollary 1.3 *For every $U \in \mathcal{N}(E)$ and the polar $U^\circ = \{f \in E' : |f(\eta)| \leq 1, \forall \eta \in U\}$,*

$$\{h \circ f : h \in \mathcal{L}_{\gamma,1}(\mathbb{C}, \mathbb{C}), f \in U^\circ\} \subset E^{(\gamma, U)}, \quad \forall \gamma \in C(0),$$

$$\{h \circ f : h \in \mathcal{H}_{\gamma,1}(\mathbb{C}, \mathbb{C}), f \in U^\circ\} \subset E^{[\gamma, U]}, \quad \forall \gamma \in C(0).$$

Proof. U° is equicontinuous [3, p.129]. \square

Theorem 1.6 *Let $\gamma_1 \in C(0)$ for which $\sup_{|t| \leq 1} |\gamma_1(t)| = 1$, $|\gamma_1(\alpha)| \leq |\gamma_1(\beta)|$ whenever $|\alpha| \leq |\beta| \leq 1$, e.g., $\gamma_1(t) = \sqrt{|t|}$. For every $U \in \mathcal{N}(E)$, $f \in E^{[\gamma_1, U]}$ and $\varepsilon > 0$ there is a $V \in \mathcal{N}(E)$ such that $\gamma_1 \circ \gamma_1 \in C(0)$ and*

$$h \circ f \in E^{(\gamma_1 \circ \gamma_1, V)}, \quad \forall h \in \mathcal{L}_{\gamma_1, \varepsilon}(\mathbb{C}, \mathbb{C}),$$

$$h \circ f \in E^{[\gamma_1 \circ \gamma_1, V]}, \quad \forall h \in \mathcal{K}_{\gamma_1, \varepsilon}(\mathbb{C}, \mathbb{C}).$$

Proof. Pick a $W \in \mathcal{N}(E)$ for which $|f(\eta)| < \varepsilon$, $\forall \eta \in W$. Let $h \in \mathcal{L}_{\gamma_1, \varepsilon}(\mathbb{C}, \mathbb{C})$, $\xi \in E$, $\eta \in V = U \cap W$ and $|t| \leq 1$. Then

$$\begin{aligned} (h \circ f)(\xi + t\eta) &= h(f(\xi + t\eta)) = h(f(\xi) + \alpha f(\eta)) \quad (|\alpha| \leq |\gamma_1(t)| \leq 1) \\ &= rh(f(\xi)) + sh(f(\eta)) = r(h \circ f)(\xi) + s(h \circ f)(\eta), \end{aligned}$$

where $|r - 1| \leq |\gamma_1(\alpha)| \leq |\gamma_1(\gamma_1(t))|$, $|s| \leq |\gamma_1(\alpha)| \leq |\gamma_1(\gamma_1(t))|$.

As in the proof of Th. 1.5, h is continuous and so $h \circ f \in E^{(\gamma_1 \circ \gamma_1, V)}$.

Similarly, $h \circ f \in E^{[\gamma_1 \circ \gamma_1, V]}$ whenever $h \in \mathcal{K}_{\gamma_1, \varepsilon}(\mathbb{C}, \mathbb{C})$. \square

Example 1.2 (1) Let $h(z) = |z|$, $\forall z \in \mathbb{C}$. Then $h \in \mathcal{K}_{\gamma_0, \mathbb{C}}(\mathbb{C}, \mathbb{C})$ where $\gamma_0(t) = t$. Let $U \in \mathcal{N}(E)$ and $\gamma_1 \in C(0)$ as in Th. 1.6. Then for every $f \in E^{(\gamma_1, U)}$ and $a > 0$ there is a $V_a \in \mathcal{N}(E)$ such that $V_a \subset U$ and $|f(\eta)| < a$, $\forall \eta \in V_a$.

Let $a > 0$, $\xi \in E$, $\eta \in V_a$ and $|t| \leq 1$. Then

$$|f(\xi + t\eta)| = |rf(\xi) + \alpha f(\eta)| = |r||f(\xi)| + |s||f(\eta)|,$$

where $||r| - 1| \leq |r - 1| \leq |\gamma_1(t)|$, $|s| \leq |\alpha| \leq |\gamma_1(t)|$. Thus, $\gamma_0 \circ \gamma_1 = \gamma_1$ and $|f(\cdot)| = h \circ f \in E^{(\gamma_1, V_a)}$, $\forall a > 0$.

(2) Let $\gamma_1(t) = \sqrt{|t|}$, $\gamma_2(t) = \frac{\pi}{2}t$, $\forall t \in \mathbb{C}$. Let $U \in \mathcal{N}(E)$ and $f \in E^{[\gamma_1, U]}$. There is a $V \in \mathcal{N}(E)$ such that $V \subset U$ and $|f(\eta)| < 1$, $\forall \eta \in V$. Define $\sin |f(\cdot)| : E \rightarrow \mathbb{C}$ by $\sin |f(\cdot)|(\xi) = \sin |f(\xi)|$, $\xi \in E$. For $\xi \in E$, $\eta \in V$ and $|t| \leq 1$,

$$\begin{aligned} \sin |f(\cdot)|(\xi + t\eta) &= \sin |f(\xi + t\eta)| = \sin |f(\xi) + \alpha f(\eta)| \quad (|\alpha| \leq |\gamma_1(t)| = \sqrt{|t|} \leq 1) \\ &= \sin[|f(\xi)| + \beta |f(\eta)|] \quad (|\beta| \leq |\alpha| \leq 1) \\ &= \sin |f(\xi)| + s \sin |f(\eta)| \quad (|s| \leq \frac{\pi}{2}|\beta| \leq \frac{\pi}{2}|\alpha| \leq \frac{\pi}{2}\sqrt{|t|} = |(\gamma_2 \circ \gamma_1)(t)|) \\ &= \sin |f(\cdot)|(\xi) + s \sin |f(\cdot)|(\eta). \end{aligned}$$

Thus, $\gamma_2 \circ \gamma_1 \in C(0)$ and $\sin |f(\cdot)| \in E^{[\gamma_2 \circ \gamma_1, V]}$.

(3) If $h(z) = e^{|z|} - 1$, $\forall z \in \mathbb{C}$, $\gamma_1(t) = \sqrt{|t|}$ and $\gamma(t) = e^2 t$, then $h \in \mathcal{L}_{\gamma, 1}(\mathbb{C}, \mathbb{C})$ and for every $f \in E^{[\gamma_1, U]}$ there is a $V \in \mathcal{N}(E)$ such that $e^{|f(\cdot)|} - 1 = h \circ f \in E^{(\gamma \circ \gamma_1, V)}$.

Even f is a nonzero usual distribution, each of $|f(\cdot)|$, $\sin |f(\cdot)|$ and $e^{|f(\cdot)|} - 1$ can not be a usual distribution. However, Th. 1.6 shows that the family of demi-distributions is closed with respect to infinitely many of nonlinear transformations.

Henceforth, in the notations $E^{(\gamma, U)}$ and $E^{[\gamma, U]}$ we always confess that $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$.

Definition 1.2 $f_k \xrightarrow{w*} f$ in $E^{(\gamma, U)}$ means that $f_k, f \in E^{(\gamma, U)}$ for all $k \in \mathbb{N}$ and $f_k(\xi) \rightarrow f(\xi)$ at each $\xi \in E$, and $f_k \rightarrow f$ in $E^{(\gamma, U)}$ means that $f_k, f \in E^{(\gamma, U)}$ for all $k \in \mathbb{N}$ and for every bounded $B \subset E$, $\lim_k f_k(\xi) = f(\xi)$ uniformly for $\xi \in B$.

Now Th. 1.2 gives the following

Theorem 1.7 $f_k \rightarrow f$ in $E^{(\gamma, U)}$ if and only if $f_k \xrightarrow{w*} f$ in $E^{(\gamma, U)}$.

Definition 1.3 A sequence $\{f_k\} \subset E^{(\gamma, U)}$ (resp., $E^{[\gamma, U]}$) is w^* Cauchy if $\lim_k f_k(\xi)$ exists at each $\xi \in E$. $E^{(\gamma, U)}$ (resp., $E^{[\gamma, U]}$) is said to be sequentially complete if for every w^* Cauchy sequence $\{f_k\}$ in $E^{(\gamma, U)}$ (resp., $E^{[\gamma, U]}$) there exists $f \in E^{(\gamma, U)}$ (resp., $E^{[\gamma, U]}$) such that $f_k \rightarrow f$, i.e., for every bounded $B \subset E$, $\lim_k f_k(\xi) = f(\xi)$ uniformly for $\xi \in B$.

Theorem 1.8 Both $\mathcal{D}_a^{(\gamma, U)}$ and $\mathcal{S}^{(\gamma, V)}$ are sequentially complete for every $\gamma \in C(0)$, $U \in \mathcal{N}(\mathcal{D}_a)$ and $V \in \mathcal{N}(\mathcal{S})$. Moreover, $\mathcal{D}^{[\gamma_0, W]}$ is also sequentially complete for $\gamma_0(t) = t$ and $W \in \mathcal{N}(\mathcal{D})$.

Proof. Let $E \in \{\mathcal{D}_a, \mathcal{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. If $\{f_k\} \subset E^{(\gamma, U)}$ and $\lim_k f_k(\xi) = f(\xi)$ exists at each $\xi \in E$, then $\{f_k\}$ is equicontinuous by Th. 3.1 of [1]. If $\xi_\nu \rightarrow \xi$ in E , then $\lim_\nu f_k(\xi_\nu) = f_k(\xi)$ uniformly for $k \in \mathbb{N}$ and $\lim_\nu f(\xi_\nu) = \lim_\nu \lim_k f_k(\xi_\nu) = \lim_k \lim_\nu f_k(\xi_\nu) = \lim_k f_k(\xi) = f(\xi)$. Thus, $f : E \rightarrow \mathbb{C}$ is continuous.

Let $\xi \in E$, $\eta \in U$ and $|t| \leq 1$. Then $f(\xi + t\eta) = \lim_k f_k(\xi + t\eta) = \lim_k [r_k f_k(\xi) + s_k f_k(\eta)]$ where $|r_k - 1| \leq |\gamma(t)|$, $|s_k| \leq |\gamma(t)|$. By passing to a subsequence if necessary, we assume that $r_k \rightarrow r$ and $s_k \rightarrow s$. Then $|r - 1| \leq |\gamma(t)|$, $|s| \leq |\gamma(t)|$ and $f(\xi + t\eta) = \lim_k [r_k f_k(\xi) + s_k f_k(\eta)] = r f(\xi) + s f(\eta)$, $f \in \mathcal{L}_{\gamma, U}(E, \mathbb{C})$. Thus, $f \in E^{(\gamma, U)}$.

Now $f_k \xrightarrow{w^*} f$ in $E^{(\gamma, U)}$ and so $f_k \rightarrow f$ in $E^{(\gamma, U)}$ by Th. 1.7.

The completeness of $\mathcal{D}^{[\gamma_0, W]}$ follows from Th. 1.4. \square

For $E \in \{\mathcal{D}, \mathcal{S}\}$ and $G \subset \mathbb{R}^n$, let $E_G = \{\xi \in E : \text{supp } \xi \subset G\}$. Each $f \in E^{(\gamma, U)}$ yields $f|_{E_G} : E_G \rightarrow \mathbb{C}$ by $f|_{E_G}(\xi) = f(\xi)$, $\forall \xi \in E_G$. Then $E_\emptyset = \{0\}$ and $f|_{E_\emptyset} = 0$, $\forall f \in E^{(\gamma, U)}$.

Definition 1.4 $E = \mathcal{D}$. For $f \in E^{(\gamma, U)}$ let

$$\text{supp } f = \mathbb{R}^n \setminus \left[\bigcup \{G \subset \mathbb{R}^n : G \text{ is open, } f|_{E_G} = 0\} \right].$$

Theorem 1.9 $E = \mathcal{D}$. For every $f \in E^{(\gamma, U)}$ there is an open $G_0 \subset \mathbb{R}^n$ such that $f|_{E_{G_0}} = 0$ and $\text{supp } f = \mathbb{R}^n \setminus G_0$.

Proof. Let $G_0 = \mathbb{R}^n \setminus \text{supp } f$ and $\{G_\alpha : \alpha \in I\} = \{G \subset \mathbb{R}^n : G \text{ is open, } f|_{E_G} = 0\}$. Then $G_0 = \bigcup_{\alpha \in I} G_\alpha$ is open and $\text{supp } f = \mathbb{R}^n \setminus G_0$.

Suppose that $f|_{E_{G_0}} \neq 0$. There is a $\xi \in E$ such that $\text{supp } \xi \subset G_0$ but $f(\xi) \neq 0$. Then $\mathbb{R}^n \setminus \text{supp } \xi \supset \mathbb{R}^n \setminus G_0 = \text{supp } f$ and $\mathbb{R}^n = (\mathbb{R}^n \setminus \text{supp } \xi) \cup (\bigcup_{\alpha \in I} G_\alpha)$. By the partition of unity, there is a sequence $\{\xi_k\} \subset \mathcal{D}$ such that $\sum_{k=1}^\infty \xi_k(x) = 1$ for all $x \in \mathbb{R}^n$ and each $\text{supp } \xi_k \subset (\mathbb{R}^n \setminus \text{supp } \xi)$ or some G_α , and each $x \in \text{supp } \xi$ has a neighborhood which intersects finitely many of $\text{supp } \xi_k$ only. But $\text{supp } \xi$ is compact and so there is an open $G \subset \mathbb{R}^n$ such that $\text{supp } \xi \subset G$ and G intersects finitely many of $\text{supp } \xi_k$ only. Hence, $\xi \xi_k = 0$ for all but finitely many of k 's. Say that $\{k : \xi \xi_k \neq 0\} = \{1, 2, \dots, m\}$. Then $\xi(x) = \sum_{k=1}^m \xi(x) \xi_k(x)$, $\forall x \in \mathbb{R}^n$. For $k \leq m$, $\xi \xi_k \neq 0$ shows that $\text{supp } \xi_k \not\subset \mathbb{R}^n \setminus \text{supp } \xi$ and so $\text{supp } \xi_k \subset G_\alpha$ for some $\alpha \in I$. Thus, $f(\xi \xi_k) = 0$, $k = 1, 2, \dots, m$.

Pick a $p \in \mathbb{N}$ such that $\frac{1}{p} \xi \xi_k \in U$, $k = 1, 2, \dots, m$. Since $\text{supp}(\frac{1}{p} \xi \xi_k) \subset$

$\text{supp } \xi_k \subset G_\alpha$ for some $\alpha \in I$, $f(\frac{1}{p}\xi\xi_k) = 0$, $k = 1, 2, \dots, m$. Then

$$\begin{aligned}
f(\xi) &= f\left(\sum_{k=1}^m \xi\xi_k\right) = f\left(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m + \frac{1}{p}\xi\xi_m\right) \\
&= r_1 f\left(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m\right) + s_1 f\left(\frac{1}{p}\xi\xi_m\right) \quad (|r_1 - 1| \leq |\gamma(1)|, |s_1| \leq |\gamma(1)|) \\
&= r_1 f\left(\sum_{k=1}^{m-1} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_m\right) \\
&\quad \dots \dots \\
&= r_1 r_2 \dots r_p f\left(\sum_{k=1}^{m-1} \xi\xi_k\right) \\
&= r_1 r_2 \dots r_p f\left(\sum_{k=1}^{m-2} \xi\xi_k + (p-1)\frac{1}{p}\xi\xi_{m-1} + \frac{1}{p}\xi\xi_{m-1}\right) \\
&= r_1 \dots r_p r_{p+1} \dots r_{2p} f\left(\sum_{k=1}^{m-2} \xi\xi_k\right) \\
&\quad \dots \dots \\
&= \left(\prod_{\nu=1}^{mp-1} r_\nu\right) f\left(\frac{1}{p}\xi\xi_1\right) = 0.
\end{aligned}$$

This contradicts that $f(\xi) \neq 0$. Hence, $f|_{E_{G_0}} = 0$. \square

Corollary 1.4 For $E = \mathcal{D}$ and $f \in E^{(\gamma, U)}$, $f = 0$ if and only if $\text{supp } f = \emptyset$.

Proof. If $f = 0$, then $f|_{E_{\mathbb{R}^n}} = 0$ and $\text{supp } f \subset \mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$, i.e., $\text{supp } f = \emptyset$.

By Th. 1.9, $\text{supp } f = \mathbb{R}^n \setminus G_0$ for some open $G_0 \subset \mathbb{R}^n$ and $f|_{E_{G_0}} = 0$. If $\text{supp } f = \emptyset$, then $\emptyset = \mathbb{R}^n \setminus G_0$ and so $G_0 = \mathbb{R}^n$ and $f|_{E_{\mathbb{R}^n}} = 0$, i.e., $f = f|_{E_{\mathbb{R}^n}} = 0$. \square

Corollary 1.5 Let $f, g \in E^{(\gamma, U)}$. Then $f = g$ if and only if for every $x \in \mathbb{R}^n$ there is a neighborhood G of x such that $f|_{E_G} = g|_{E_G}$.

Proof. If for every $x \in \mathbb{R}^n$ there is an open $G_x \subset \mathbb{R}^n$ such that $x \in G_x$ and $f|_{E_{G_x}} = g|_{E_{G_x}}$, i.e., $(f - g)|_{E_{G_x}} = 0$, then $\text{supp } (f - g) \subset \mathbb{R}^n \setminus (\bigcup_{x \in \mathbb{R}^n} G_x) = \emptyset$ and so $f - g = 0$. \square

2 Differentiation

$$E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}, [E^{(\gamma, U)}] = \text{span}(E^{(\gamma, U)}) \text{ in } \mathbb{C}^E.$$

Definition 2.1 Let $f = \sum_{k=1}^m \alpha_k f_k \in [E^{(\gamma, U)}]$ where each $\alpha_k \in \mathbb{C}$, $f_k \in E^{(\gamma, U)}$. Observing each $\xi \in E$ is a function defined on \mathbb{R}^n , for $j \in \{1, 2, \dots, n\}$ define $\frac{\partial f}{\partial x_j} : E \rightarrow \mathbb{C}$ by

$$\frac{\partial f}{\partial x_j}(\xi) = f\left(-\frac{\partial \xi}{\partial x_j}\right), \quad \xi \in E.$$

Then $\frac{\partial}{\partial x_j}(\sum_{k=1}^m \alpha_k f_k)(\xi) = (\sum_{k=1}^m \alpha_k f_k)(-\frac{\partial \xi}{\partial x_j}) = \sum_{k=1}^m \alpha_k f_k(-\frac{\partial \xi}{\partial x_j}) = \sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j}(\xi) = (\sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j})(\xi)$ and so $\frac{\partial}{\partial x_j}(\sum_{k=1}^m \alpha_k f_k) = \sum_{k=1}^m \alpha_k \frac{\partial f_k}{\partial x_j}$ for $\sum_{k=1}^m \alpha_k f_k \in [E^{(\gamma, U)}]$.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ let $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$. As in the case of usual distributions, for $f \in [E^{(\gamma, U)}]$ and every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, $(D^\alpha f)(\xi) = f((-1)^{|\alpha|} D^\alpha \xi)$, $\forall \xi \in E$. Evidently, we have that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^5 f}{\partial x_1 \partial x_2^2 \partial x_3^2} = \frac{\partial^5 f}{\partial x_3^2 \partial x_1 \partial x_2^2}$, etc.

Lemma 2.1 For every multi-index α , $D^\alpha : E \rightarrow E$ is a continuous linear operator.

Proof. For $E = \mathcal{D}_a$ or \mathcal{S} , the conclusion is obvious.

Let $\xi_k \rightarrow 0$ in \mathcal{D} . Then $\{\xi_k\} \subset \mathcal{D}_m$ for some $m \in \mathbb{N}$ and $\xi_k \rightarrow 0$ in \mathcal{D}_m since \mathcal{D}_m is a subspace of \mathcal{D} [3, p.219]. Then $\|D^\alpha \xi_k\|_p \leq \|\xi_k\|_{|\alpha|+p}$ for all $p \in \mathbb{N}$ and so $D^\alpha \xi_k \rightarrow 0$ in \mathcal{D}_m , i.e., $D^\alpha \xi_k \rightarrow 0$ in \mathcal{D} . Thus, $D^\alpha : \mathcal{D} \rightarrow \mathcal{D}$ is sequentially continuous. Then $D^\alpha : \mathcal{D} \rightarrow \mathcal{D}$ is continuous since \mathcal{D} is bornological and C -sequential. See also Th. 1.1. \square

Theorem 2.1 Let α be a multi-index. For every $U \in \mathcal{N}(E)$ there is a $V \in \mathcal{N}(E)$ such that

$$\begin{aligned} \{D^\alpha f : f \in E^{(\gamma, U)}\} &\subset E^{(\gamma, V)}, \quad \forall \gamma \in C(0), \\ \{D^\alpha f : f \in [E^{(\gamma, U)}]\} &\subset [E^{(\gamma, V)}], \quad \forall \gamma \in C(0). \end{aligned}$$

Moreover, if $f_k, f \in E^{(\gamma, U)}$ and $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \rightarrow f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_k (D^\alpha f_k)(\xi) = (D^\alpha f)(\xi)$ uniformly for $\xi \in B$.

Proof. Let $U \in \mathcal{N}(E)$. By Lemma 2.1, there is a $V \in \mathcal{N}(E)$ for which $(-1)^{|\alpha|} D^\alpha \eta \in U$, $\forall \eta \in V$.

Let $f \in E^{(\gamma, U)}$, $\xi \in E$, $\eta \in V$ and $|t| \leq 1$. Then

$$\begin{aligned} (D^\alpha f)(\xi + t\eta) &= f((-1)^{|\alpha|} D^\alpha \xi + t(-1)^{|\alpha|} D^\alpha \eta) \\ &= rf((-1)^{|\alpha|} D^\alpha \xi) + sf((-1)^{|\alpha|} D^\alpha \eta) \\ &= r(D^\alpha f)(\xi) + s(D^\alpha f)(\eta), \end{aligned}$$

where $|r - 1| \leq |\gamma(t)|$, $|s| \leq |\gamma(t)|$. Thus, $D^\alpha f \in \mathcal{L}_{\gamma, V}(E, \mathbb{C})$.

Since both $(-1)^{|\alpha|} D^\alpha : E \rightarrow E$ and $f : E \rightarrow \mathbb{C}$ are continuous, $D^\alpha f = f \circ (-1)^{|\alpha|} D^\alpha : E \rightarrow \mathbb{C}$ is also continuous and so $D^\alpha f \in E^{(\gamma, V)}$.

Suppose that $f_k, f \in E^{(\gamma, U)}$, $f_k(\xi) \rightarrow f(\xi)$, $\forall \xi \in E$, and $B \subset E$ is bounded. Then $(-1)^{|\alpha|} D^\alpha(B) = \{(-1)^{|\alpha|} D^\alpha \xi : \xi \in B\}$ is bounded and, by Th. 1.7, $\lim_k (D^\alpha f_k)(\xi) = \lim_k f_k((-1)^{|\alpha|} D^\alpha \xi) = f((-1)^{|\alpha|} D^\alpha \xi) = (D^\alpha f)(\xi)$ uniformly for $\xi \in B$. \square

Corollary 2.1 Let α be a multi-index and $U \in \mathcal{N}(E)$. There is a $V \in \mathcal{N}(E)$ such that for every $\gamma \in C(0)$ the differentiation operator $D^\alpha : [E^{(\gamma, U)}] \rightarrow [E^{(\gamma, V)}]$ is linear and $w* - w*$ continuous.

Proof. For $f, g \in [E^{(\gamma, U)}]$ and $t \in \mathbb{C}$, $(D^\alpha(f + tg))(\xi) = (f + tg)((-1)^{|\alpha|} D^\alpha \xi) = f((-1)^{|\alpha|} D^\alpha \xi) + tg((-1)^{|\alpha|} D^\alpha \xi) = (D^\alpha f)(\xi) + t(D^\alpha g)(\xi)$, $\forall \xi \in E$, i.e., $D^\alpha(f + tg) = D^\alpha f + tD^\alpha g$.

If $f_k, f \in [E^{(\gamma, U)}]$ such that $f_k \xrightarrow{w*} f$, then

$$(D^\alpha f_k)(\xi) = f_k((-1)^{|\alpha|} D^\alpha \xi) \rightarrow f((-1)^{|\alpha|} D^\alpha \xi) = (D^\alpha f)(\xi), \quad \forall \xi \in E. \quad \square$$

Example 2.1 (1) Let $f \in L_{loc}^1(\mathbb{R}^n)$, $\gamma \in C(0)$ and

$$[f](\xi) = \int_{\mathbb{R}^n} |f(x)\xi(x)| dx, \quad \xi \in \mathcal{D}.$$

Then $[f] \in \mathcal{D}^{[\gamma, \mathcal{D}]}$ (see Exam. 1.1(1)), and

$$(D^\alpha[f])(\xi) = \int_{\mathbb{R}^n} |f(x)(D^\alpha\xi)(x)| dx, \quad \forall \xi \in \mathcal{D}.$$

(2) γ and U as in Exam. 1.1(2), and

$$f(\xi) = \int_{-\infty}^{\infty} |\sin \xi(x)| dx, \quad \forall \xi \in \mathcal{D}_1(\mathbb{R}).$$

Then $f \in (\mathcal{D}_1(\mathbb{R}))^{[\gamma, U]}$ and

$$\begin{aligned} (D^\alpha f)(\xi) &= \int_{-\infty}^{\infty} |\sin[(-1)^{|\alpha|}(D^\alpha\xi)(x)]| dx \\ &= \int_{-\infty}^{\infty} |\sin(D^\alpha\xi)(x)| dx, \quad \forall \xi \in \mathcal{D}_1(\mathbb{R}). \end{aligned}$$

(3) Let $U \in \mathcal{N}(E)$ and $\gamma(t) = \sqrt{|t|}$, $\forall t \in \mathbb{C}$. For every $f \in E^{[\gamma, U]}$, both $\sin|f(\cdot)|$ and $e^{|f(\cdot)|} - 1$ are demi-distributions, see Exam. 1.2. Then for every multi-index α ,

$$D^\alpha \sin|f(\cdot)| = \sin|D^\alpha f(\cdot)|, \quad D^\alpha(e^{|f(\cdot)|} - 1) = e^{|D^\alpha f(\cdot)|} - 1.$$

In general, Th. 1.6 shows that $h \circ f$ is a demi-distribution for every $h \in \mathcal{L}_{\gamma, \varepsilon}(\mathbb{C}, \mathbb{C})$. Then

$$D^\alpha(h \circ f) = h \circ D^\alpha f.$$

In fact, $(D^\alpha(h \circ f))(\xi) = (h \circ f)((-1)^{|\alpha|} D^\alpha \xi) = h[f((-1)^{|\alpha|} D^\alpha \xi)] = h[(D^\alpha f)(\xi)] = (h \circ D^\alpha f)(\xi)$, $\forall \xi \in E$.

(4) $E \in \{\mathcal{D}_a, \mathcal{S}\}$ and $\{\|\cdot\|_p\}_{p=0}^\infty$ is the usual norm sequence on E . For $p \in \mathbb{N}$ and $\varepsilon > 0$, let $U_{p, \varepsilon} = \{\eta \in E : \|\eta\|_p < \varepsilon\}$. Then for every multi-index α and $\varepsilon > 0$,

$$D^\alpha f \in E^{(\gamma, U_{p+|\alpha|, \varepsilon})}, \quad \forall f \in E^{(\gamma, U_{p, \varepsilon})}, \quad \gamma \in C(0), \quad p \in \mathbb{N},$$

$$D^\alpha f \in [E^{(\gamma, U_{p+|\alpha|, \varepsilon})}], \quad \forall f \in [E^{(\gamma, U_{p, \varepsilon})}], \quad \gamma \in C(0), \quad p \in \mathbb{N}.$$

In fact, $\|\eta\|_{p+|\alpha|} < \varepsilon$ implies $\|(-1)^{|\alpha|} D^\alpha \eta\|_p \leq \|\eta\|_{p+|\alpha|} < \varepsilon$.

Definition 2.2 $\zeta : \mathbb{R}^n \rightarrow \mathbb{C}$ is called a multiplier in E if for every $\xi \in E$ the pointwise product $\zeta\xi \in E$ and $\zeta\xi_k \rightarrow 0$ in E whenever $\xi_k \rightarrow 0$. For a multiplier ζ in E and $f \in [E^{(\gamma, U)}]$, define $\zeta f : E \rightarrow \mathbb{C}$ by $(\zeta f)(\xi) = f(\zeta\xi)$, $\forall \xi \in E$.

Theorem 2.2 If ζ is a multiplier in E , then for every $U \in \mathcal{N}(E)$ there is a $V \in \mathcal{N}(E)$ such that

$$\{\zeta f : f \in E^{(\gamma, U)}\} \subset E^{(\gamma, V)}, \quad \forall \gamma \in C(0),$$

$$\{\zeta f : f \in [E^{(\gamma, U)}]\} \subset [E^{(\gamma, V)}], \quad \forall \gamma \in C(0).$$

Proof. The correspondence $\xi \mapsto \zeta\xi$ is a continuous linear operator from E into E and so there is a $V \in \mathcal{N}(E)$ such that $\zeta\eta \in U$ for all $\eta \in V$.

Let $f \in E^{(\gamma, U)}$ and $\xi \in E$, $\eta \in V$, $|t| \leq 1$. Then

$$\begin{aligned} (\zeta f)(\xi + t\eta) &= f(\zeta\xi + t\zeta\eta) = rf(\zeta\xi) + sf(\zeta\eta) \\ &= r(\zeta f)(\xi) + s(\zeta f)(\eta), \quad |r - 1| \leq |\gamma(t)|, \quad |s| \leq |\gamma(t)|. \end{aligned}$$

Thus, $\zeta f \in \mathcal{L}_{\gamma, V}(E, \mathbb{C})$. The continuity of $\zeta f : E \rightarrow \mathbb{C}$ follows from the continuity of f and the continuity of the correspondence $\xi \mapsto \zeta\xi$. Hence, $\zeta f \in E^{(\gamma, V)}$. \square

Lemma 2.2 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R})\}$ be a space of test functions defined on \mathbb{R} , i.e., $n = 1$. Pick a $\zeta \in E$ for which $\int_{-\infty}^{\infty} \zeta(x) dx = 1$ and define $A : E \rightarrow E$ by $A(\xi) = \xi - (\int_{-\infty}^{\infty} \xi(x) dx)\zeta$, $\xi \in E$. Then A is a continuous linear operator, $\int_{-\infty}^{\infty} A(\xi)(x) dx = 0$ for all $\xi \in E$ and $A(\xi^{(k)}) = \xi^{(k)}$, $\forall \xi \in E$, $k \in \mathbb{N}$.

Proof. For $\xi, \eta \in E$ and $t \in \mathbb{C}$, $A(\xi + t\eta) = \xi + t\eta - (\int_{-\infty}^{\infty} (\xi + t\eta)(x) dx)\zeta = \xi - (\int_{-\infty}^{\infty} \xi(x) dx)\zeta + t\eta - t(\int_{-\infty}^{\infty} \eta(x) dx)\zeta = A(\xi) + tA(\eta)$.

Since $1 \in E'$, if $\xi_k \rightarrow \xi$ in E , then $\int_{-\infty}^{\infty} \xi_k(x) dx \rightarrow \int_{-\infty}^{\infty} \xi(x) dx$, $A(\xi_k) = \xi_k - (\int_{-\infty}^{\infty} \xi_k(x) dx)\zeta \rightarrow \xi - (\int_{-\infty}^{\infty} \xi(x) dx)\zeta = A(\xi)$ and so A is sequentially continuous. Since E is bornological, A is continuous.

For $\xi \in E$ and $k \geq 1$, $A(\xi^{(k)}) = \xi^{(k)} - (\int_{-\infty}^{\infty} \xi^{(k)}(x) dx)\zeta = \xi^{(k)}$. \square

For usual distributions, the equation $y' = 0$ has solutions $y = \text{const}$ only. However, for demi-distributions in $E^{(\gamma, U)}$, the equation $y' = 0$ has extremely many solutions which are not constants, and the equation $y' = f$ also has extremely many solutions which are demi-distributions.

Lemma 2.3 Let E be a space of test functions defined on \mathbb{R} . Let $\gamma \in C(0)$ and $U \in \mathcal{N}(E)$. For $y \in E^{(\gamma, U)}$, $y' = 0$ if and only if $y(\xi) = 0$ whenever $\int_{-\infty}^{\infty} \xi(x) dx = 0$.

Proof. Suppose that $y' = 0$ and $\xi \in E$ for which $\int_{-\infty}^{\infty} \xi(x) dx = 0$. Letting $\eta(x) = \int_{-\infty}^x \xi(t) dt$, $\eta \in E$ and $\xi = \eta'$. Then $y(\xi) = y(-(-\eta)') = y'(-\eta) = 0$.

The converse is obvious. \square

Example 2.2 (1) Let $\gamma \in C(0)$ and $U \in \mathcal{N}(\mathcal{D}(\mathbb{R}))$. Pick a nonzero $\xi_0 \in \mathcal{D}(\mathbb{R})$, e.g., $\xi_0(x) = e^{1/(x^2-1)}$ if $|x| < 1$ and 0 if $|x| \geq 1$. Since the functional $\xi \mapsto \int_{-\infty}^{\infty} \xi(x) dx$ is continuous on $\mathcal{D}(\mathbb{R})$, i.e., $1 \in (\mathcal{D}(\mathbb{R}))'$, there is a $V \in \mathcal{N}(\mathcal{D}(\mathbb{R}))$ such that $(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0 \in U$, $\forall \eta \in V$. Then pick a nonzero $f_0 \in (\mathcal{D}(\mathbb{R}))^{(\gamma, U)}$ and define $f : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ by $f(\xi) = f_0((\int_{-\infty}^{\infty} \xi(x) dx)\xi_0)$, $\xi \in \mathcal{D}(\mathbb{R})$.

Evidently, f is continuous. For $\xi \in \mathcal{D}(\mathbb{R})$, $\eta \in V$ and $|t| \leq 1$,

$$\begin{aligned} f(\xi + t\eta) &= f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0 + t(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0] \\ &= rf_0((\int_{-\infty}^{\infty} \xi(x) dx)\xi_0) + sf_0((\int_{-\infty}^{\infty} \eta(x) dx)\xi_0) \\ &= rf(\xi) + sf(\eta), \quad |r - 1| \leq |\gamma(t)|, \quad |s| \leq |\gamma(t)|. \end{aligned}$$

Thus, $f \in (\mathcal{D}(\mathbb{R}))^{(\gamma, V)}$.

If $\xi \in E$ and $\int_{-\infty}^{\infty} \xi(x) dx = 0$, then $f(\xi) = f_0(0) = 0$ so $f' = 0$ by Lemma 2.3, i.e., f is a solution of the equation $y' = 0$. If $\xi_0(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$ and $f_0 = [1]$, i.e., $f_0(\xi) = \int_{-\infty}^{\infty} |\xi(x)| dx$, $\forall \xi \in \mathcal{D}(\mathbb{R})$, then

$$\begin{aligned} f(\xi) &= f_0\left[\left(\int_{-\infty}^{\infty} \xi(x) dx\right)\xi_0\right] = \int_{-\infty}^{\infty} \left|\left(\int_{-\infty}^{\infty} \xi(t) dt\right)\xi_0(x)\right| dx \\ &= \left|\int_{-\infty}^{\infty} \xi(x) dx\right| \int_{-1}^1 e^{1/(x^2-1)} dx, \quad \forall \xi \in \mathcal{D}(\mathbb{R}). \end{aligned}$$

Note that this f is not a usual distribution because

$$\begin{aligned} f(\xi + \eta) &= \left|\int_{-\infty}^{\infty} \xi(x) dx + \int_{-\infty}^{\infty} \eta(x) dx\right| \int_{-\infty}^{\infty} e^{1/(x^2-1)} dx \\ &\neq \left[\left|\int_{-\infty}^{\infty} \xi(x) dx\right| + \left|\int_{-\infty}^{\infty} \eta(x) dx\right|\right] \int_{-\infty}^{\infty} e^{1/(x^2-1)} dx \\ &= f(\xi) + f(\eta) \end{aligned}$$

for many pairs $\xi, \eta \in \mathcal{D}(\mathbb{R})$, e.g., if $\xi = \xi_0$, $\eta = -\xi_0$, then $f(\xi + \eta) = f(0) = 0$ but $f(\xi) + f(\eta) = 2\left[\int_{-1}^1 e^{1/(x^2-1)} dx\right]^2 > 0$. But $f' = 0$.

The solution $f(\xi) = f_0\left[\left(\int_{-\infty}^{\infty} \xi(x) dx\right)\xi_0\right] = \left|\int_{-\infty}^{\infty} \xi(x) dx\right| \int_{-\infty}^{\infty} |\xi_0(x)| dx$ determined by $\xi_0 \in E$ and $f_0 = [1]$ has a nice splitting property: $f \in (\mathcal{D}(\mathbb{R}))^{[\gamma_0, \mathcal{D}(\mathbb{R})]}$. In fact, for $\xi, \eta \in \mathcal{D}(\mathbb{R})$ and $t \in \mathbb{R}$,

$$\begin{aligned} f(\xi + t\eta) &= \left|\int_{-\infty}^{\infty} [\xi(x) + t\eta(x)] dx\right| \int_{-\infty}^{\infty} |\xi_0(x)| dx \\ &= \left|\int_{-\infty}^{\infty} \xi(x) dx + t \int_{-\infty}^{\infty} \eta(x) dx\right| \int_{-\infty}^{\infty} |\xi_0(x)| dx \\ &= \left[\left|\int_{-\infty}^{\infty} \xi(x) dx\right| + s \left|\int_{-\infty}^{\infty} \eta(x) dx\right|\right] \int_{-\infty}^{\infty} |\xi_0(x)| dx \\ &= f(\xi) + sf(\eta), \quad |s| \leq |t|. \end{aligned}$$

(2) Let $\gamma(t) = \frac{\pi}{2}t$ for $t \in \mathbb{R}$, $U = \{\xi \in \mathcal{D}_1(\mathbb{R}) : \max_{|x| \leq 1} |\xi(x)| < 1\}$ and

$$f_0(\xi) = \int_{-\infty}^{\infty} |\sin \xi(x)| dx, \quad \xi \in \mathcal{D}_1(\mathbb{R}).$$

Then $f_0 \in (\mathcal{D}_1(\mathbb{R}))^{[\gamma, U]}$ (Exam. 1.1(2)).

Pick a nonzero $\xi_0 \in \mathcal{D}_1(\mathbb{R})$ and let $f(\xi) = f_0\left[\left(\int_{-\infty}^{\infty} \xi(x) dx\right)\xi_0\right] = \int_{-\infty}^{\infty} |\sin[(\int_{-\infty}^{\infty} \xi(t) dt)\xi_0(x)]| dx$ for $\xi \in \mathcal{D}_1(\mathbb{R})$. Then pick a $V \in \mathcal{N}(\mathcal{D}_1(\mathbb{R}))$ such that $(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0 \in U$, $\forall \eta \in V$.

Let $\xi \in \mathcal{D}_1(\mathbb{R})$, $\eta \in V$ and $|t| \leq 1$. Then $\max_{|x| \leq 1} |(\int_{-\infty}^{\infty} \eta(s) ds)\xi_0(x)| \leq 1$, and

$$\begin{aligned} f(\xi + t\eta) &= \int_{-\infty}^{\infty} \left|\sin\left[\left(\int_{-\infty}^{\infty} \xi(r) dr\right)\xi_0(x) + t\left(\int_{-\infty}^{\infty} \eta(r) dr\right)\xi_0(x)\right]\right| dx \\ &= \int_{-\infty}^{\infty} \left|\sin\left[\left(\int_{-\infty}^{\infty} \xi(r) dr\right)\xi_0(x)\right] + \alpha(x) \sin\left[\left(\int_{-\infty}^{\infty} \eta(r) dr\right)\xi_0(x)\right]\right| dx \\ &= \int_{-\infty}^{\infty} \left|\sin\left[\left(\int_{-\infty}^{\infty} \xi(r) dr\right)\xi_0(x)\right]\right| dx + \int_{-\infty}^{\infty} \beta(x) \left|\sin\left[\left(\int_{-\infty}^{\infty} \eta(r) dr\right)\xi_0(x)\right]\right| dx, \end{aligned}$$

where $|\beta(x)| \leq |\alpha(x)| \leq \frac{\pi}{2}|t|$ for all $x \in \mathbb{R}$. It is similar to Exam. 1.1,

$$\begin{aligned} f(\xi + t\eta) &= \int_{-\infty}^{\infty} |\sin[(\int_{-\infty}^{\infty} \xi(r) dr)\xi_0(x)]| dx + s \int_{-\infty}^{\infty} |\sin[(\int_{-\infty}^{\infty} \eta(r) dr)\xi_0(x)]| dx \\ &= f(\xi) + sf(\eta), \quad |s| \leq \frac{\pi}{2}|t| = |\gamma(t)|. \end{aligned}$$

Then $f \in (\mathcal{D}_1(\mathbb{R}))^{[\gamma, V]}$ and $f' = 0$.

In general, we have

Theorem 2.3 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R})\}$, a space of test functions defined on \mathbb{R} . Let $U \in \mathcal{N}(E)$, $\gamma \in C(0)$. Then for every $\xi_0 \in E$ and $f_0 \in E^{(\gamma, U)}$ there is a $V \in \mathcal{N}(E)$ such that the equation $y' = 0$ has a solution $f \in E^{(\gamma, V)}$ which is given by $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0]$ for $\xi \in E$. If $f_0 = 1$, then $f_0 \in E'$ and $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0] = (\int_{-\infty}^{\infty} \xi(x) dx)f_0(\xi_0) = (\int_{-\infty}^{\infty} \xi(x) dx)(\int_{-\infty}^{\infty} \xi_0(\tau) d\tau) = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} \xi_0(\tau) d\tau)\xi(x) dx$, $\forall \xi \in E$, i.e., $f = \int_{-\infty}^{\infty} \xi_0(\tau) d\tau \in E'$, a constant which is a usual solution of the equation $y' = 0$.

The solutions of the equation $y' = 0$ have an interesting property as follows.

Theorem 2.4 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$, a space of test functions defined on \mathbb{R} . Let $U \in \mathcal{N}(E)$, $\gamma \in C(0)$ and $y \in E^{[\gamma, U]}$. If $y' = 0$, then for every $\zeta \in E$ with $\int_{-\infty}^{\infty} \zeta(x) dx = 1$,

$$y(\xi) = y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta], \quad \forall \xi \in E,$$

i.e., if $\zeta_1, \zeta_2 \in E$ such that $\int_{-\infty}^{\infty} \zeta_1(x) dx = \int_{-\infty}^{\infty} \zeta_2(x) dx = 1$, then $y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta_1] = y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta_2] = y(\xi)$ for all $\xi \in E$ and, in particular,

$$y(\xi) = y(\eta) \text{ whenever } \int_{-\infty}^{\infty} \xi(x) dx = \int_{-\infty}^{\infty} \eta(x) dx = 1,$$

$$y\left(\frac{\xi}{\int_{-\infty}^{\infty} \xi(x) dx}\right) = y\left(\frac{\eta}{\int_{-\infty}^{\infty} \eta(x) dx}\right) \text{ whenever } \int_{-\infty}^{\infty} \xi(x) dx \neq 0 \text{ and } \int_{-\infty}^{\infty} \eta(x) dx \neq 0.$$

Proof. If $\zeta \in E$ such that $\zeta \neq \xi'$, $\forall \xi \in E$, i.e., $\int_{-\infty}^{\infty} \zeta(x) dx \neq 0$, then $\frac{1}{\int_{-\infty}^{\infty} \zeta(x) dx}\zeta \in E$ and $\int_{-\infty}^{\infty} \frac{\zeta(x)}{\int_{-\infty}^{\infty} \zeta(t) dt} dx = 1$. Pick a $\zeta \in E$ for which $\int_{-\infty}^{\infty} \zeta(x) dx = 1$ and let $A(\xi) = \xi - (\int_{-\infty}^{\infty} \xi(x) dx)\zeta$ for $\xi \in E$. By Lemma 2.2, $A : E \rightarrow E$ is a continuous linear operator and $\int_{-\infty}^{\infty} A(\xi)(x) dx = 0$, $\forall \xi \in E$. Moreover,

$$A(\xi)(x) = \left(\int_{-\infty}^x A(\xi)(t) dt \right)', \quad \forall \xi \in E, \quad x \in \mathbb{R}.$$

Let $\xi \in E$ and pick a $p \in \mathbb{N}$ for which $\frac{1}{p}A(\xi) \in U$. Then

$$\begin{aligned}
y(\xi) &= y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta + A(\xi)] \\
&= y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta + (p-1)\frac{1}{p}A(\xi) + \frac{1}{p}A(\xi)] \\
&= y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta + (p-1)\frac{1}{p}A(\xi)] + s_1 y(\frac{1}{p}A(\xi)) \\
&\quad \dots \\
&= y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta] + sy(\frac{1}{p}A(\xi)).
\end{aligned}$$

But $\frac{1}{p}A(\xi)(x) = (\frac{1}{p} \int_{-\infty}^x A(\xi)(t) dt)'$ for all $x \in \mathbb{R}$ and so $y(\frac{1}{p}A(\xi)) = y[-(-\frac{1}{p} \int_{-\infty}^x A(\xi)(t) dt)'] = y'(-\frac{1}{p} \int_{-\infty}^x A(\xi)(t) dt) = 0$ since $y' = 0$. Therefore,

$$y(\xi) = y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta], \quad \forall \xi \in E.$$

If $\int_{-\infty}^{\infty} \xi(x) dx = \int_{-\infty}^{\infty} \eta(x) dx = 1$, then $y(\xi) = y[(\int_{-\infty}^{\infty} \xi(x) dx)\zeta] = y(\zeta) = y[(\int_{-\infty}^{\infty} \eta(x) dx)\zeta] = y(\eta)$. \square

For $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$ let

$$E_1 = \{\xi \in E : \int_{-\infty}^{\infty} \xi(x) dx = 1\}.$$

If $y \in E'$ is a usual distribution such that $y' = 0$, then y must be a constant $C \in \mathbb{R}$, i.e., $y(\xi) = \int_{-\infty}^{\infty} C\xi(x) dx$ for all $\xi \in E$. Hence, $y(\xi) = C$, $\forall \xi \in E_1$. Th. 2.3 shows that the same fact holds for the case of $E^{[\gamma, U]}$.

Corollary 2.2 $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. If $y \in E^{[\gamma, U]}$ such that $y' = 0$, then $y(\cdot)$ is an invariant on E_1 , i.e., there is a $C \in \mathbb{R}$ such that

$$y(\xi) = C, \quad \forall \xi \in E_1.$$

Although Th. 2.3 gives a lot of various solutions of the equation $y' = 0$ for the case of $E^{(\gamma, U)}$ but Th. 2.3 did not give all solutions. However, for the case of $E^{[\gamma, U]}$ we can give all solutions of $y' = 0$.

Theorem 2.5 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$ be a space of test functions defined on \mathbb{R} , $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. Then for every $\xi_0 \in E$ and $f_0 \in E^{[\gamma, U]}$ there is a $V \in \mathcal{N}(E)$ such that the equation $y' = 0$ has a solution $f \in E^{[\gamma, V]}$ which is given by $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0]$ for $\xi \in E$. Conversely, if $f \in E^{[\gamma, U]}$ is a solution of the equation $y' = 0$, then there exist $\xi_0 \in E$ and $f_0 \in E^{[\gamma, U]}$ such that $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0]$, $\forall \xi \in E$.

Proof. Let $\xi_0 \in E$, $f_0 \in E^{[\gamma, U]}$. There is a $V \in \mathcal{N}(E)$ such that $(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0 \in U$ for all $\eta \in V$. Let $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0]$ for $\xi \in E$. If $\xi \in E$, $\eta \in V$ and $|t| \leq 1$, then $f(\xi + t\eta) = f_0[(\int_{-\infty}^{\infty} (\xi + t\eta)(x) dx)\xi_0] = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0 +$

$t(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0] = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0] + sf_0[(\int_{-\infty}^{\infty} \eta(x) dx)\xi_0] = f(\xi) + sf(\eta)$,
where $|s| \leq |\gamma(t)|$. Thus, $f \in E^{[\gamma, V]}$ and $f' = 0$:

$$f'(\xi) = f(-\xi') = f_0[(\int_{-\infty}^{\infty} -\xi'(x) dx)\xi_0] = f_0(0) = 0, \forall \xi \in E.$$

Conversely, suppose that $f \in E^{[\gamma, U]}$ and $f' = 0$. Pick a $\xi_0 \in E$ with $\int_{-\infty}^{\infty} \xi_0(x) dx = 1$, and let $f_0 = f$. By Th. 2.4,

$$f(\xi) = f[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0] = f_0[(\int_{-\infty}^{\infty} \xi(x) dx)\xi_0], \forall \xi \in E. \square$$

We now consider the equation $y' = f$ where $f \in E^{(\gamma, U)}$.

Theorem 2.6 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$ be a space of test functions defined on \mathbb{R} , $E_1 = \{\xi \in E : \int_{-\infty}^{\infty} \xi(x) dx = 1\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. Let $f \in E^{(\gamma, U)}$ be arbitrary. Then every $\zeta \in E_1$ gives $U_\zeta \in \mathcal{N}(E)$ and $y_\zeta \in E^{(\gamma, U_\zeta)}$ such that $y'_\zeta = f$ and

$$y_\zeta(\xi) = f(-\int_{-\infty}^x [\xi(\tau) - (\int_{-\infty}^{\infty} \xi(s) ds)\zeta(\tau)] d\tau), \forall \xi \in E.$$

Proof. Only need to consider $E = \mathcal{D}(\mathbb{R})$. Pick a $\zeta \in E_1$ and define $A_\zeta : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}(\mathbb{R})$ by $A_\zeta(\xi) = \xi - (\int_{-\infty}^{\infty} \xi(\tau) d\tau)\zeta$, $\forall \xi \in \mathcal{D}(\mathbb{R})$. By Lemma 2.2, A_ζ is a continuous linear operator and $\int_{-\infty}^{\infty} A_\zeta(\xi)(\tau) d\tau = 0$, $\forall \xi \in \mathcal{D}(\mathbb{R})$. For every $\xi \in \mathcal{D}(\mathbb{R})$, $A_\zeta(\xi)(x) = \frac{d}{dx}[\int_{-\infty}^x A_\zeta(\xi)(\tau) d\tau]$, $\forall x \in \mathbb{R}$. Since $A_\zeta(\xi) \in \mathcal{D}(\mathbb{R})$, $\forall \xi \in \mathcal{D}(\mathbb{R})$, there is an $a_\xi > 0$ such that $A_\zeta(\xi)(x) = 0$, $\forall |x| > a_\xi$ and so $\int_{-\infty}^x A_\zeta(\xi)(\tau) d\tau = 0$ for $x < -a_\xi$ and $\int_{-\infty}^x A_\zeta(\xi)(\tau) d\tau = \int_{-\infty}^{\infty} A_\zeta(\xi)(\tau) d\tau = 0$ for $x > a_\xi$. Thus, $\int_{-\infty}^x A_\zeta(\xi)(\tau) d\tau$ gives a test function in $\mathcal{D}(\mathbb{R})$, $\forall \xi \in \mathcal{D}(\mathbb{R})$.

Let $T(\xi)(x) = \int_{-\infty}^x A_\zeta(\xi)(\tau) d\tau$, $\forall \xi \in \mathcal{D}(\mathbb{R})$, $x \in \mathbb{R}$. Since A_ζ is linear, $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}(\mathbb{R})$ is a linear operator. Let $\xi_k \rightarrow 0$ in $\mathcal{D}(\mathbb{R})$. By Lemma 2.2, $A_\zeta(\xi_k) \rightarrow 0$ in $\mathcal{D}(\mathbb{R})$ and so $A_\zeta(\xi_k) \rightarrow 0$ in $\mathcal{D}_{m_0}(\mathbb{R})$ for some $m_0 \in \mathbb{N}$ [3, p.219]. Then $\{T(\xi_k)\} \subset \mathcal{D}_{m_0}(\mathbb{R})$. In fact, $\{A_\zeta(\xi_k)\} \subset \mathcal{D}_{m_0}(\mathbb{R})$, i.e., $A_\zeta(\xi_k)(x) = 0$, $\forall |x| > m_0$, $k \in \mathbb{N}$, hence $T(\xi_k)(x) = \int_{-\infty}^x A_\zeta(\xi_k)(\tau) d\tau = 0$ for $x < -m_0$, $k \in \mathbb{N}$ and $T(\xi_k)(x) = \int_{-\infty}^x A_\zeta(\xi_k)(\tau) d\tau = \int_{-\infty}^{\infty} A_\zeta(\xi_k)(\tau) d\tau = 0$ whenever $x > m_0$ and $k \in \mathbb{N}$. Since $A_\zeta(\xi_k) \rightarrow 0$ in $\mathcal{D}_{m_0}(\mathbb{R})$,

$$\begin{aligned} \|T(\xi_k)\|_0 &= \sup_{|x| \leq m_0} |T(\xi_k)(x)| = \sup_{|x| \leq m_0} |\int_{-m_0}^x A_\zeta(\xi_k)(\tau) d\tau| \\ &\leq \int_{-m_0}^{m_0} |A_\zeta(\xi_k)(\tau)| d\tau \leq 2m_0 \|A_\zeta(\xi_k)\|_0 \rightarrow 0, \text{ i.e., } \|T(\xi_k)\|_0 \rightarrow 0. \end{aligned}$$

Moreover, $\frac{dT(\xi)}{dx}(x) = A_\zeta(\xi)(x)$, $\forall \xi \in \mathcal{D}(\mathbb{R})$, $x \in \mathbb{R}$, i.e., $\frac{dT(\xi)}{dx} = A_\zeta(\xi)$, $\forall \xi \in \mathcal{D}(\mathbb{R})$. Since $\{T(\xi_k)\} \subset \mathcal{D}_{m_0}(\mathbb{R})$ and $A_\zeta(\xi_k) \rightarrow 0$ in $\mathcal{D}_{m_0}(\mathbb{R})$, $\|T(\xi_k)\|_p \leq \max\{\|T(\xi_k)\|_0, \|A_\zeta(\xi_k)\|_{p-1}\} \rightarrow 0$ for each $p \in \mathbb{N}$. Thus, $T(\xi_k) \rightarrow 0$ in $\mathcal{D}_{m_0}(\mathbb{R})$, i.e., $T(\xi_k) \rightarrow 0$ in $\mathcal{D}(\mathbb{R})$. Therefore, $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}(\mathbb{R})$ is sequentially continuous. Since T is linear and $\mathcal{D}(\mathbb{R})$ is bornological, i.e., $\mathcal{D}(\mathbb{R})$ is C -sequential, T is continuous and so there is a balanced $U_\zeta \in \mathcal{N}(\mathcal{D}(\mathbb{R}))$ such that $T(U_\zeta) \subset U$.

Define $y_\zeta : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ by

$$y_\zeta(\xi) = f(-T(\xi)) = f(-\int_{-\infty}^x [\xi(\tau) - (\int_{-\infty}^{\infty} \xi(s) ds)\zeta(\tau)] d\tau), \forall \xi \in \mathcal{D}(\mathbb{R}).$$

Since both f and T are continuous, y_ζ is continuous.

Let $\xi \in \mathcal{D}(\mathbb{R})$, $\eta \in U_\zeta$ and $|t| \leq 1$. Since U_ζ is balanced and $f \in \mathcal{D}(\mathbb{R})^{(\gamma, U)}$, $T(-\eta) \in T(U_\zeta) \subset U$ and

$$\begin{aligned} y_\zeta(\xi + t\eta) &= f(-T(\xi + t\eta)) = f(-T(\xi) + tT(-\eta)) = rf(-T(\xi)) + sf(T(-\eta)) \\ &= rf(-T(\xi)) + sf(-T(\eta)) = ry_\zeta(\xi) + sy_\zeta(\eta), \quad |r - 1| \leq |\gamma(t)|, |s| \leq |\gamma(t)|. \end{aligned}$$

Thus, $y_\zeta \in \mathcal{D}(\mathbb{R})^{(\gamma, U_\zeta)}$.

For every $\xi \in \mathcal{D}(\mathbb{R})$, $T(\xi')(x) = \int_{-\infty}^x A_\zeta(\xi')(\tau) d\tau = \int_{-\infty}^x [\xi'(\tau) - (\int_{-\infty}^\infty \xi'(s) ds)\zeta(\tau)] d\tau = \int_{-\infty}^x \xi'(\tau) d\tau = \xi(x)$, $\forall x \in \mathbb{R}$, i.e., $T(\xi') = \xi$, $\forall \xi \in \mathcal{D}(\mathbb{R})$. Then

$$y'_\zeta(\xi) = y_\zeta(-\xi') = f(-T(-\xi')) = f(T(\xi')) = f(\xi), \quad \forall \xi \in \mathcal{D}(\mathbb{R}),$$

i.e., $y'_\zeta = f$. \square

Corollary 2.3 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$ and $f \in E^{(\gamma, U)}$. Then every $\zeta \in E$ which $\zeta \neq \xi'$ for all $\xi \in E$ (i.e., $\int_{-\infty}^\infty \zeta(x) dx \neq 0$) gives a balanced $U_\zeta \in \mathcal{N}(E)$ and a solution y_ζ of the equation $y' = f$ such that $y_\zeta \in E^{(\gamma, U_\zeta)}$ and

$$y_\zeta(\xi) = f\left(-\int_{-\infty}^x \left[\xi(\tau) - \frac{\int_{-\infty}^\infty \xi(s) ds}{\int_{-\infty}^\infty \zeta(s) ds} \zeta(\tau)\right] d\tau\right), \quad \forall \xi \in E.$$

Theorem 2.7 Let $E \in \{\mathcal{D}_a(\mathbb{R}), \mathcal{D}(\mathbb{R})\}$ and $f \in E^{(\gamma, U)}$. For every $\xi_0 \in E$, $f_0 \in E^{(\gamma, U)}$ and $\zeta \in E$ with $\int_{-\infty}^\infty \zeta(x) dx \neq 0$ let

$$g(\xi) = f_0\left[\left(\int_{-\infty}^\infty \xi(x) dx\right)\xi_0\right] + f\left(-\int_{-\infty}^x \left[\xi(\tau) - \frac{\int_{-\infty}^\infty \xi(s) ds}{\int_{-\infty}^\infty \zeta(s) ds} \zeta(\tau)\right] d\tau\right), \quad \forall \xi \in E,$$

then $g \in [E^{(\gamma, W)}]$ for some $W \in \mathcal{N}(E)$ and $g' = f$.

Proof. Let $\xi_0 \in E$, $f_0 \in E^{(\gamma, U)}$ and $\zeta \in E$ with $\int_{-\infty}^\infty \zeta(x) dx \neq 0$. Let

$$g_0(\xi) = f_0\left[\left(\int_{-\infty}^\infty \xi(x) dx\right)\xi_0\right], \quad \xi \in E,$$

$$y_\zeta(\xi) = f\left(-\int_{-\infty}^x \left[\xi(\tau) - \frac{\int_{-\infty}^\infty \xi(s) ds}{\int_{-\infty}^\infty \zeta(s) ds} \zeta(\tau)\right] d\tau\right), \quad \xi \in E.$$

Then $g_0 \in E^{(\gamma, V)}$ for some $V \in \mathcal{N}(E)$ and $g'_0 = 0$ by Th. 2.3, and $y_\zeta \in E^{(\gamma, U_\zeta)}$ for some balanced $U_\zeta \in \mathcal{N}(E)$ and $y'_\zeta = f$ by Cor. 2.3.

Let $W = V \cap U_\zeta$. Then $W \in \mathcal{N}(E)$ and $E^{(\gamma, V)} \cup E^{(\gamma, U_\zeta)} \subset E^{(\gamma, W)}$. Thus, $g_0 \in E^{(\gamma, W)}$, $y_\zeta \in E^{(\gamma, W)}$, $g = g_0 + y_\zeta \in [E^{(\gamma, W)}]$ and $g' = (g_0 + y_\zeta)' = g'_0 + y'_\zeta = f$. \square

Further discussions of ordinary and partial differential equations of demi-distributions will be interesting but we reserve these discussions for another paper.

3 Fourier Transform

Let $x + iy = (x_1 + iy_1, \dots, x_n + iy_n) \in \mathbb{C}^n$, $|y| = |y_1| + \dots + |y_n|$. For $a > 0$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, a multi-index, let $(x + iy)^\alpha = \prod_{k=1}^n (x_k + iy_k)^{\alpha_k}$ and

$$Z(a) = \{\zeta \in \mathbb{C}^{\mathbb{C}^n} : \zeta \text{ is entire; for every multi-index } \alpha, |(x+iy)^\alpha \zeta(x+iy)| \leq C_\alpha(\zeta) e^{a|y|}\},$$

$$\|\zeta\|_p = \sup_{x+iy \in \mathbb{C}^n, |\alpha| \leq p} |(x+iy)^\alpha \zeta(x+iy)| e^{-a|y|}, \quad p = 0, 1, 2, 3, \dots,$$

$$Z = \{\zeta \in \mathbb{C}^{\mathbb{C}^n} : \zeta \text{ is entire, } \exists a(\zeta) > 0 \text{ such that } |(x+iy)^\alpha \zeta(x+iy)| \leq C_\alpha(\zeta) e^{a(\zeta)|y|}\}.$$

The Fourier transform $F(\xi)$ of $\xi \in \mathcal{D}$ is given by

$$F(\xi)(\sigma + i\tau) = \zeta(\sigma + i\tau) = \int \xi(x) e^{i(x, \sigma) - (x, \tau)} dx, \quad (x, \sigma) = \sum_{j=1}^n x_j \sigma_j, \quad (x, \tau) = \sum_{j=1}^n x_j \tau_j.$$

Then $F[\mathcal{D}_a] = Z_a$, $F[\mathcal{D}] = Z$, and operators

$$F : \mathcal{D}_a \rightarrow Z_a, \quad F : \mathcal{D} \rightarrow Z, \quad F^{-1} : Z_a \rightarrow \mathcal{D}_a, \quad F^{-1} : Z \rightarrow \mathcal{D}$$

are both continuous and linear [4, 3.1.1—3.1.2].

For $\xi \in \mathcal{S}$ let

$$F(\xi)(\sigma) = \zeta(\sigma) = \int \xi(x) e^{i(x, \sigma)} dx, \quad \forall \sigma \in \mathbb{R}^n.$$

Then $F(\mathcal{S}) = \mathcal{S}$ and both F and F^{-1} are continuous and linear.

Definition 3.1 Let $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. For $f \in E^{(\gamma, U)}$ define $\hat{f} : F(E) \rightarrow \mathbb{C}$ by $\hat{f}(\zeta) = (2\pi)^n f(F^{-1}(\zeta))$, $\forall \zeta \in F(E)$. We write $\hat{f} = F(f)$ and so

$$F(f)(F(\xi)) = (2\pi)^n f(\xi), \quad \forall f \in E^{(\gamma, U)}, \quad \xi \in E.$$

Henceforth, $E \in \{\mathcal{D}_a, \mathcal{D}, \mathcal{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$.

Theorem 3.1 $F(E^{(\gamma, U)}) = (F(E))^{(\gamma, F(U))}$.

Proof. Since both F and F^{-1} are continuous linear operators, $F(U) \in \mathcal{N}(F(E))$. Let $f \in E^{(\gamma, U)}$, $\zeta \in F(E)$, $\eta \in F(U)$ and $|t| \leq 1$. Then

$$\begin{aligned} F(f)(\zeta + t\eta) &= (2\pi)^n f(F^{-1}(\zeta + t\eta)) = (2\pi)^n f(F^{-1}(\zeta) + tF^{-1}(\eta)) \\ &= (2\pi)^n [rf(F^{-1}(\zeta)) + sf(F^{-1}(\eta))] \\ &= r(2\pi)^n f(F^{-1}(\zeta)) + s(2\pi)^n f(F^{-1}(\eta)) \\ &= rF(f)(\zeta) + sF(f)(\eta), \quad |r - 1| \leq |\gamma(t)|, \quad |s| \leq |\gamma(t)|. \end{aligned}$$

Thus, $F(f) \in \mathcal{L}_{\gamma, F(U)}(F(E), \mathbb{C})$.

Let $\zeta_\alpha \rightarrow \zeta$ in $F(E)$. Then $F^{-1}(\zeta_\alpha) \rightarrow F^{-1}(\zeta)$ in E and so $F(f)(\zeta_\alpha) = (2\pi)^n f(F^{-1}(\zeta_\alpha)) \rightarrow (2\pi)^n f(F^{-1}(\zeta)) = F(f)(\zeta)$. This shows that $F(f)$ is continuous and $F(f) \in (F(E))^{(\gamma, F(U))}$.

Conversely, for $g \in (F(E))^{(\gamma, F(U))}$ define

$$f(\xi) = (2\pi)^{-n} g(F(\xi)), \quad \forall \xi \in E.$$

If $\xi \in E$, $\eta \in U$ and $|t| \leq 1$, then

$$\begin{aligned} f(\xi + t\eta) &= (2\pi)^{-n} g(F(\xi + t\eta)) = (2\pi)^{-n} g(F(\xi) + tF(\eta)) \\ &= (2\pi)^{-n} [rg(F(\xi)) + sg(F(\eta))] \\ &= rf(\xi) + sf(\eta), \quad |r-1| \leq |\gamma(t)|, \quad |s| \leq |\gamma(t)|, \end{aligned}$$

i.e., $f \in \mathcal{L}_{\gamma, U}(E, U)$.

Since both g and F are continuous, f is continuous so $f \in E^{(\gamma, U)}$ and $F(f)(\zeta) = (2\pi)^n f(F^{-1}(\zeta)) = g(\zeta)$, $\forall \zeta \in F(E)$, i.e., $g = F(f)$. \square

Definition 3.2 Let $[(F(E))^{(\gamma, F(U))}] = \text{span}(F(E))^{(\gamma, F(U))}$ in $\mathbb{C}^{F(E)}$. For $f \in [E^{(\gamma, U)}]$ define $F(f) : F(E) \rightarrow \mathbb{C}$ by

$$F(f)(F(\xi)) = (2\pi)^n f(\xi), \quad \forall \xi \in E.$$

Theorem 3.2 If $f = \sum_{k=1}^m \alpha_k f_k$ where $\alpha_k \in \mathbb{C}$ and $f_k \in E^{(\gamma, E)}$, then $F(f) = \sum_{k=1}^m \alpha_k F(f_k) \in [(F(E))^{(\gamma, F(U))}]$, and

$$F([E^{(\gamma, U)}]) = [(F(E))^{(\gamma, F(U))}].$$

Moreover, $F : [E^{(\gamma, U)}] \rightarrow [(F(E))^{(\gamma, F(U))}]$ is $w * -w*$ continuous and linear.

Proof. For $\xi \in E$, $F(f)(F(\xi)) = (2\pi)^n f(\xi) = (2\pi)^n \sum_{k=1}^m \alpha_k f_k(\xi) = \sum_{k=1}^m \alpha_k F(f_k)(F(\xi)) = (\sum_{k=1}^m \alpha_k F(f_k))(F(\xi))$. Thus, $F(f) = \sum_{k=1}^m \alpha_k F(f_k) \in [(F(E))^{(\gamma, F(U))}]$.

Let $g = \sum_{k=1}^m \alpha_k g_k \in [(F(E))^{(\gamma, F(U))}]$ where $\alpha_k \in \mathbb{C}$, $g_k \in (F(E))^{(\gamma, F(U))}$. By Th. 3.1, each $g_k = F(f_k)$ for some $f_k \in E^{(\gamma, U)}$ and so $g = \sum_{k=1}^m \alpha_k g_k = \sum_{k=1}^m \alpha_k F(f_k) = F(\sum_{k=1}^m \alpha_k f_k) \in F([E^{(\gamma, U)}])$ because $(\sum_{k=1}^m \alpha_k F(f_k))(F(\xi)) = \sum_{k=1}^m \alpha_k F(f_k)(F(\xi)) = (2\pi)^n \sum_{k=1}^m \alpha_k f_k(\xi) = (2\pi)^n (\sum_{k=1}^m \alpha_k f_k)(\xi) = F(\sum_{k=1}^m \alpha_k f_k)(F(\xi))$, $\forall \xi \in E$.

If $f_\alpha \xrightarrow{w*} f$ in $[E^{(\gamma, U)}]$, i.e., $f_\alpha(\xi) \rightarrow f(\xi)$, $\forall \xi \in E$, then $F(f_\alpha)(F(\xi)) = (2\pi)^n f_\alpha(\xi) \rightarrow (2\pi)^n f(\xi) = F(f)(F(\xi))$, $\forall \xi \in E$, i.e., $F(f_\alpha) \xrightarrow{w*} F(f)$.

For $f, h \in [E^{(\gamma, U)}]$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} F(\alpha f + \beta h)(F(\xi)) &= (2\pi)^n (\alpha f + \beta h)(\xi) \\ &= (2\pi)^n (\alpha f(\xi) + \beta h(\xi)) \\ &= \alpha F(f)(F(\xi)) + \beta F(h)(F(\xi)) \\ &= (\alpha F(f) + \beta F(h))(F(\xi)), \quad \forall \xi \in E, \end{aligned}$$

i.e., $F(\alpha f + \beta h) = \alpha F(f) + \beta F(h)$. \square

Now we consider the case of $n = 1$. Let $\mathcal{S}(\mathbb{R})$ be the space of infinitely differentiable but rapidly decreasing functions defined on \mathbb{R} . Then $F(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$, $F((\mathcal{S}(\mathbb{R}))') = (\mathcal{S}(\mathbb{R}))'$.

A constant $C \in (\mathcal{S}(\mathbb{R}))'$ means that $C(\zeta) = \int_{-\infty}^{\infty} C\zeta(\sigma) d\sigma$ for all $\zeta \in \mathcal{S}(\mathbb{R})$ [4, 3.2.1], and $C = F(C\delta) = CF(\delta)$. In fact, $CF(\delta)(F(\xi)) = 2\pi C\delta(\xi) = 2\pi C\xi(0) = 2\pi C \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i0\sigma} F(\xi)(\sigma) d\sigma = \int_{-\infty}^{\infty} CF(\xi)(\sigma) d\sigma = C(F(\xi))$ for all $\xi \in \mathcal{S}(\mathbb{R})$.

Lemma 3.1 Let $y \in (\mathcal{S}(\mathbb{R}))'$, a usual distribution. Then

$$y(i\sigma\zeta(\sigma)) = 0 \text{ for all } \zeta \in \mathcal{S}(\mathbb{R})$$

if and only if $y = C\delta$, where C is a constant.

Proof. Suppose that $y \in (\mathcal{S}(\mathbb{R}))'$ and $y(i\sigma\zeta(\sigma)) = 0$, $\forall \zeta \in \mathcal{S}(\mathbb{R})$. Since $(\mathcal{S}(\mathbb{R}))' = F((\mathcal{S}(\mathbb{R}))')$, there is a usual distribution $f \in (\mathcal{S}(\mathbb{R}))'$ such that $y = F(f)$ and

$$\begin{aligned} f'(\xi) &= f(-\xi') = \frac{1}{2\pi} F(f)(F((- \xi)')) = \frac{1}{2\pi} F(f)(-i\sigma F(-\xi)(\sigma)) \\ &= \frac{1}{2\pi} y(i\sigma F(\xi)(\sigma)) = 0, \quad \forall \xi \in \mathcal{S}(\mathbb{R}), \end{aligned}$$

i.e., $f' = 0$. But f is a usual distribution so $f = C$, a constant. Thus, $y = F(f) = F(C) = CF(1) = C\delta$.

Conversely, if $y = C\delta$ where C is a constant, then

$$y(i\sigma\zeta(\sigma)) = C\delta(i\sigma\zeta(\sigma)) = 0, \quad \forall \zeta \in \mathcal{S}(\mathbb{R}). \quad \square$$

However, there exists a lot of various demi-distributions on $\mathcal{S}(\mathbb{R})$ which satisfy the condition $y(i\sigma\zeta(\sigma)) = 0$, $\forall \zeta \in \mathcal{S}(\mathbb{R})$.

Theorem 3.3 Let $U \in \mathcal{N}(\mathcal{S}(\mathbb{R}))$ and $\gamma \in C(0)$. Pick an arbitrary $f_0 \in (\mathcal{S}(\mathbb{R}))^{(\gamma, U)}$ and $\xi_0 \in \mathcal{S}(\mathbb{R})$ and let $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(t) dt)\xi_0]$, $\forall \xi \in \mathcal{S}(\mathbb{R})$. Then $f \in (\mathcal{S}(\mathbb{R}))^{(\gamma, V)}$ for some $V \in \mathcal{N}(\mathcal{S}(\mathbb{R}))$ and $F(f) \in (\mathcal{S}(\mathbb{R}))^{(\gamma, F(V))}$ such that

$$F(f)(i\sigma\zeta(\sigma)) = 0, \quad \forall \zeta \in \mathcal{S}(\mathbb{R}).$$

Proof. By Th. 2.3, there is a $V \in \mathcal{N}(\mathcal{S}(\mathbb{R}))$ such that $f \in (\mathcal{S}(\mathbb{R}))^{(\gamma, V)}$ and $f' = 0$. If $\zeta \in \mathcal{S}(\mathbb{R})$, then $\zeta = F(\xi)$ for some $\xi \in \mathcal{S}(\mathbb{R})$ and

$$\begin{aligned} F(f)(i\sigma\zeta(\sigma)) &= F(f)(i\sigma F(\xi)(\sigma)) = F(f)(-i\sigma F(-\xi)(\sigma)) \\ &= F(f)(F((- \xi)')) = 2\pi f(-\xi') = 2\pi f'(\xi) = 0. \quad \square \end{aligned}$$

Example 3.1 Let $\gamma(t) = \frac{\pi}{2}t$ for $t \in \mathbb{R}$, $U = \{\eta \in \mathcal{S}(\mathbb{R}) : |\eta(0)| < 1\}$. Then $f_0 = \sin \circ \delta \in (\mathcal{S}(\mathbb{R}))^{(\gamma, U)}$, where $f_0(\xi) = \sin[\delta(\xi)] = \sin \xi(0)$ for all $\xi \in \mathcal{S}(\mathbb{R})$.

Let $\xi_0 = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}$ and $f(\xi) = f_0[(\int_{-\infty}^{\infty} \xi(t) dt)\xi_0]$ for $\xi \in \mathcal{S}(\mathbb{R})$, i.e.,

$f(\xi) = \sin[\delta((\int_{-\infty}^{\infty} \xi(t) dt)\xi_0)] = \sin(e^{-1} \int_{-\infty}^{\infty} \xi(t) dt)$, $\forall \xi \in \mathcal{S}(\mathbb{R})$. Pick a $V \in \mathcal{N}(\mathcal{S}(\mathbb{R}))$ for which $(\int_{-\infty}^{\infty} \eta(t) dt)\xi_0 \in U$, $\forall \eta \in V$. Then $f \in (\mathcal{S}(\mathbb{R}))^{(\gamma, V)}$ and $f'(\xi) = f(-\xi') = \sin(-e^{-1} \int_{-\infty}^{\infty} \xi'(t) dt) = 0$ for all $\xi \in \mathcal{S}(\mathbb{R})$, i.e., $f' = 0$. Therefore, $F(f)(i\sigma\zeta(\sigma)) = 0$, $\forall \zeta \in \mathcal{S}(\mathbb{R})$.

By Th. 3.1, $F(f) \in (\mathcal{S}(\mathbb{R}))^{(\gamma, F(V))}$ and

$$F(f)(\zeta) = 2\pi f(F^{-1}(\zeta)) = 2\pi \sin[e^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\sigma} \zeta(\sigma) d\sigma dx], \quad \forall \zeta \in \mathcal{S}(\mathbb{R}),$$

$$F(f)(F(\xi)) = 2\pi f(\xi) = 2\pi \sin[e^{-1} \int_{-\infty}^{\infty} \xi(x) dx], \quad \forall \xi \in \mathcal{S}(\mathbb{R}).$$

Since $C\delta(F(\xi)) = C\delta(\int_{-\infty}^{\infty} e^{ix\sigma} \xi(x) dx) = C \int_{-\infty}^{\infty} \xi(x) dx$ for all $\xi \in \mathcal{S}(\mathbb{R})$, $F(f) \neq C\delta$ for every constant C .

4 Convolutions

In this section, $E \in \{\mathcal{D}, \mathcal{S}\}$, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$.

Definition 4.1 ([4, 3.3.2]) A distribution $f_0 \in E'$ is called a convolution multiplier on E if the following (i) and (ii) hold for f_0 :

(i) if for each $\xi \in E$ define $f_0 * \xi : \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$(f_0 * \xi)(x) = f_0(\xi(x + \cdot)), \quad \forall x \in \mathbb{R}^n,$$

then $f_0 * \xi \in E$;

(ii) if $\xi_k \rightarrow 0$ in E , then $f_0 * \xi_k \rightarrow 0$ in E .

Lemma 4.1 If f_0 is a convolution multiplier on E , then $f_0 * \cdot : E \rightarrow E$ is a continuous linear operator.

Proof. For $\xi, \eta \in E$ and $t \in \mathbb{C}$, $(f_0 * (\xi + t\eta))(x) = f_0(\xi(x + \cdot) + t\eta(x + \cdot)) = f_0(\xi(x + \cdot)) + tf_0(\eta(x + \cdot)) = (f_0 * \xi)(x) + t(f_0 * \eta)(x)$, $\forall x \in \mathbb{R}^n$, i.e., $f_0 * (\xi + t\eta) = f_0 * \xi + t(f_0 * \eta)$.

By Def. 4.1, $f_0 * \cdot$ is sequentially continuous. Since \mathcal{S} is metric and \mathcal{D} is bornological, $f_0 * \cdot$ is continuous. \square

Following [4], $P(D) = \sum a_\alpha D^\alpha = \sum a_{\alpha_1, \dots, \alpha_n} \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ is a finite sum, where $a_\alpha \in \mathbb{C}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index. For $\xi \in E$ and $x \in \mathbb{R}^n$, $\frac{\partial \xi(x + \tau)}{\partial \tau_j} = \frac{\partial \xi(x + \tau)}{\partial (x_j + \tau_j)} \frac{\partial (x_j + \tau_j)}{\partial \tau_j} = \frac{\partial \xi(x + \tau)}{\partial (x_j + \tau_j)} (\frac{\partial x_j}{\partial \tau_j} + \frac{\partial \tau_j}{\partial \tau_j}) = \frac{\partial \xi(x + \tau)}{\partial (x_j + \tau_j)}$ and, by induction, it is easy to see that $\frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial \tau_1^{\alpha_1} \dots \partial \tau_n^{\alpha_n}} = \frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial (x_1 + \tau_1)^{\alpha_1} \dots \partial (x_n + \tau_n)^{\alpha_n}}$ for every multi-index α and so there is no any ambiguity for the notation $D^\alpha \xi(x + \cdot)$, i.e.,

$$D^\alpha \xi(x + \cdot) = \frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial \tau_1^{\alpha_1} \dots \partial \tau_n^{\alpha_n}} = \frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial (x_1 + \tau_1)^{\alpha_1} \dots \partial (x_n + \tau_n)^{\alpha_n}} = (D^\alpha \xi)(x + \cdot).$$

Theorem 4.1 If $f_0 \in E'$ is a convolution multiplier on E , then $P(D)f_0$ is also a convolution multiplier on E , and

$$(\sum a_\alpha D^\alpha f_0) * \xi = \sum a_\alpha (-1)^{|\alpha|} f_0 * D^\alpha \xi, \quad \forall \xi \in E.$$

Proof. For $\xi \in E$, define $(\sum a_\alpha D^\alpha f_0) * \xi : \mathbb{R}^n \rightarrow \mathbb{C}$ by

$$((\sum a_\alpha D^\alpha f_0) * \xi)(x) = (\sum a_\alpha D^\alpha f_0)(\xi(x + \cdot)) = \sum a_\alpha (D^\alpha f_0)(\xi(x + \cdot)), \quad \forall x \in \mathbb{R}^n.$$

For $x \in \mathbb{R}^n$ we write $\xi(x + \tau) = \zeta_x(\tau)$, $\forall \tau \in \mathbb{R}^n$. Then $\zeta_x \in E$ and $(D^\alpha f_0)(\xi(x + \cdot)) = (D^\alpha f_0)(\zeta_x) = f_0((-1)^{|\alpha|} D^\alpha \zeta_x) = f_0((-1)^{|\alpha|} \frac{\partial^{|\alpha|} \zeta_x}{\partial \tau_1^{\alpha_1} \dots \partial \tau_n^{\alpha_n}}) = f_0((-1)^{|\alpha|} \frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial \tau_1^{\alpha_1} \dots \partial \tau_n^{\alpha_n}}) = f_0[(-1)^{|\alpha|} (D^\alpha \xi)(x + \cdot)],$

$$\begin{aligned} ((\sum a_\alpha D^\alpha f_0) * \xi)(x) &= \sum a_\alpha (D^\alpha f_0)(\xi(x + \cdot)) \\ &= \sum a_\alpha f_0[(-1)^{|\alpha|} (D^\alpha \xi)(x + \cdot)] \\ &= \sum a_\alpha (-1)^{|\alpha|} f_0[(D^\alpha \xi)(x + \cdot)] \\ &= \sum a_\alpha (-1)^{|\alpha|} (f_0 * D^\alpha \xi)(x) \\ &= (\sum a_\alpha (-1)^{|\alpha|} f_0 * D^\alpha \xi)(x), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Thus, $(\sum a_\alpha D^\alpha f_0) * \xi = \sum a_\alpha (-1)^{|\alpha|} f_0 * D^\alpha \xi \in E$, $\forall \xi \in E$.
Let $\xi_k \rightarrow 0$ in E . By Lemma 2.1, $D^\alpha \xi_k \rightarrow 0$ in E and

$$\lim_k (\sum a_\alpha D^\alpha f_0) * \xi_k = \sum a_\alpha (-1)^{|\alpha|} \lim_k (f_0 * D^\alpha \xi_k) = 0. \quad \square$$

Definition 4.2 For every convolution multiplier $f_0 \in E'$ and $f \in [E^{(\gamma, U)}]$ define the convolution $f_0 * f : E \rightarrow \mathbb{C}$ by

$$(f_0 * f)(\xi) = f(f_0 * \xi), \quad \forall \xi \in E.$$

Example 4.1 (1) For every $\xi \in E$, $\delta * \xi = \xi$, $(D^\alpha \delta) * \xi = (-1)^{|\alpha|} D^\alpha \xi$, and for every $f \in [E^{(\gamma, U)}]$, $\delta * f = f$, $(D^\alpha \delta) * f = D^\alpha f$.

In fact, for $\xi \in E$ and $x \in \mathbb{R}^n$, $(\delta * \xi)(x) = \delta(\xi(x + \cdot)) = \xi(x + 0) = \xi(x)$,
 $((D^\alpha \delta) * \xi)(x) = (D^\alpha \delta)(\xi(x + \cdot)) = \delta((-1)^{|\alpha|} \frac{\partial^{|\alpha|} \xi(x + \tau)}{\partial \tau_1^{\alpha_1} \dots \partial \tau_n^{\alpha_n}}) = (-1)^{|\alpha|} (D^\alpha \xi)(x)$, i.e.,
 $\delta * \xi = \xi$, $(D^\alpha \delta) * \xi = (-1)^{|\alpha|} D^\alpha \xi$. Then for every $\xi \in E$,

$$(\delta * f)(\xi) = f(\delta * \xi) = f(\xi), \quad ((D^\alpha \delta) * f)(\xi) = f((D^\alpha \delta) * \xi) = f((-1)^{|\alpha|} D^\alpha \xi) = (D^\alpha f)(\xi).$$

(2) Let $f \in E'$ and $U = \{\eta \in E : |f(\eta)| < 1\}$. By Cor. 1.1, $h \circ f \in E^{(\gamma, U)}$ for each $h \in \mathcal{L}_{\gamma, 1}(\mathbb{C}, \mathbb{C})$ and $f_0 * (h \circ f) = h \circ (f_0 * f)$ for every convolution multiplier $f_0 \in E'$. In fact,

$$(f_0 * (h \circ f))(\xi) = (h \circ f)(f_0 * \xi) = h[f(f_0 * \xi)] = h[(f_0 * f)(\xi)] = [h \circ (f_0 * f)](\xi), \quad \forall \xi \in E.$$

Theorem 4.2 Let $f_0 \in E'$ be a convolution multiplier. For every $U \in \mathcal{N}(E)$ there is a $V \in \mathcal{N}(E)$ such that

$$f_0 * f \in E^{(\gamma, V)}, \quad \forall f \in E^{(\gamma, U)},$$

$$f_0 * f \in [E^{(\gamma, V)}], \quad \forall f \in [E^{(\gamma, U)}].$$

Proof. Let $T(\xi) = f_0 * \xi$, $\forall \xi \in E$. By Lemma 4.1, T is a continuous linear operator. Let $U \in \mathcal{N}(E)$ and $f = \sum_{k=1}^m \alpha_k f_k \in [E^{(\gamma, U)}]$ where $\alpha_k \in \mathbb{C}$ and $f_k \in E^{(\gamma, U)}$. There is a $V \in \mathcal{N}(E)$ such that $T(V) \subset U$, i.e., $f_0 * \eta \in U$, $\forall \eta \in V$. Since $(f_0 * f)(\xi) = f(f_0 * \xi) = f(T(\xi))$ for all $\xi \in E$ and both f and T are continuous, $f_0 * f : E \rightarrow \mathbb{C}$ is continuous.

Let $\xi \in E$, $\eta \in V$ and $|t| \leq 1$. For $1 \leq k \leq m$,

$$\begin{aligned} (f_0 * f_k)(\xi + t\eta) &= f_k[f_0 * (\xi + t\eta)] = f_k[T(\xi + t\eta)] \\ &= f_k(T(\xi) + tT(\eta)) = r_k f_k(T(\xi)) + s_k f_k(T(\eta)) \\ &= r_k (f_0 * f_k)(\xi) + s_k (f_0 * f_k)(\eta), \end{aligned}$$

where $|r_k - 1| \leq |\gamma(t)|$, $|s_k| \leq |\gamma(t)|$. Thus, $f_0 * f_k \in E^{(\gamma, V)}$, $k = 1, 2, \dots, m$.

Now $(f_0 * f)(\xi) = f(f_0 * \xi) = (\sum_{k=1}^m \alpha_k f_k)(f_0 * \xi) = \sum_{k=1}^m \alpha_k f_k(f_0 * \xi) = \sum_{k=1}^m \alpha_k (f_0 * f_k)(\xi) = (\sum_{k=1}^m \alpha_k (f_0 * f_k))(\xi)$, $\forall \xi \in E$. This shows that $f_0 * f = \sum_{k=1}^m \alpha_k (f_0 * f_k) \in [E^{(\gamma, V)}]$. \square

Henceforth, we write $f_0 * \xi = f_0(\xi(x + \cdot)) = (f_0 * \xi)(x)$, see [4, 3.3.2].

Observe that $tf \in E^{(\gamma, U)}$ whenever $t \in \mathbb{C}$ and $f \in E^{(\gamma, U)}$.

Lemma 4.2 *If $f_0 \in E'$ is a convolution multiplier and $t \in \mathbb{C}$, then*

$$t(f_0 * \xi) = (tf_0) * \xi, \quad \forall \xi \in E;$$

$$t(f_0 * f) = f_0 * (tf), \quad \forall f \in [E^{(\gamma, U)}];$$

$$t(f_0 * f) = (tf_0) * f, \quad \forall f \in E'.$$

Proof. $t(f_0 * \xi) = tf_0(\xi(x + \cdot)) = (tf_0)(\xi(x + \cdot)) = (tf_0) * \xi, \quad \forall \xi \in E.$ For $f \in [E^{(\gamma, U)}]$ and $\xi \in E$, $t(f_0 * f)(\xi) = tf(f_0 * \xi) = (tf)(f_0 * \xi) = (f_0 * (tf))(\xi)$. If $f \in E'$, then $t(f_0 * f)(\xi) = tf(f_0 * \xi) = f(tf_0 * \xi) = f((tf_0) * \xi) = ((tf_0) * f)(\xi)$ for all $\xi \in E$. \square

As usual, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, \dots, 0, 1)$.

Theorem 4.3 *If $f_0 \in E'$ is a convolution multiplier and α is a multi-index, then $D^\alpha(f_0 * \xi) = f_0 * D^\alpha \xi$ for $\xi \in E$, and*

$$D^\alpha(f_0 * f) = (D^\alpha f_0) * f = f_0 * D^\alpha f, \quad \forall f \in [E^{(\gamma, U)}].$$

Proof. As in [4], for $\xi \in E$ and $1 \leq j \leq n$ we have $x + he_j = (x_1, \dots, x_n) + (0, \dots, 0, \overset{(j)}{h}, 0, \dots, 0)$ and $\frac{\partial(f_0 * \xi)}{\partial x_j} = \lim_{h \rightarrow 0} \frac{1}{h} [(f_0 * \xi)(x + he_j) - (f_0 * \xi)(x)] = \lim_{h \rightarrow 0} \frac{1}{h} [f_0(\xi(x + he_j + \cdot)) - f_0(\xi(x + \cdot))] = \lim_{h \rightarrow 0} f_0\left(\frac{\xi(x + he_j + \cdot) - \xi(x + \cdot)}{h}\right) = f_0\left(\lim_{h \rightarrow 0} \frac{\xi(x + (\cdot + he_j)) - \xi(x + \cdot)}{h}\right) = f_0\left(\frac{\partial \xi}{\partial \tau_j}(x + \tau)\right) = f_0((D^{e_j} \xi)(x + \cdot)) = f_0 * D^{e_j} \xi = f_0 * \frac{\partial \xi}{\partial x_j}$, i.e., $\frac{\partial(f_0 * \xi)}{\partial x_j} = f_0 * \frac{\partial \xi}{\partial x_j}$ [4, Th. 3.3.3]. If $D^\alpha(f_0 * \xi) = f_0 * D^\alpha \xi$, then $D^{\alpha+e_j}(f_0 * \xi) = \frac{\partial}{\partial x_j}(D^\alpha(f_0 * \xi)) = \frac{\partial}{\partial x_j}(f_0 * D^\alpha \xi) = f_0 * \frac{\partial D^\alpha \xi}{\partial x_j} = f_0 * D^{\alpha+e_j} \xi$. Thus, $D^\alpha(f_0 * \xi) = f_0 * D^\alpha \xi$ for every multi-index α .

Let $f \in [E^{(\gamma, U)}]$, $\xi \in E$ and $1 \leq j \leq n$. By Th. 4.2, $f_0 * f \in [E^{(\gamma, V)}]$ for some $V \in \mathcal{N}(E)$ and $\frac{\partial(f_0 * f)}{\partial x_j}(\xi) = (f_0 * f)\left(-\frac{\partial \xi}{\partial x_j}\right) = f(f_0 * (-1)\frac{\partial \xi}{\partial x_j}) = f[f_0(-\frac{\partial \xi}{\partial \tau_j}(x + \tau))] = f[\frac{\partial f_0}{\partial \tau_j}(\xi(x + \tau))] = f((D^{e_j} f_0) * \xi) = ((D^{e_j} f_0) * f)(\xi) = (\frac{\partial f_0}{\partial x_j} * f)(\xi)$. Thus, $\frac{\partial(f_0 * f)}{\partial x_j} = \frac{\partial f_0}{\partial x_j} * f$.

Suppose that $D^\alpha(f_0 * f) = (D^\alpha f_0) * f$. Then

$$D^{\alpha+e_j}(f_0 * f) = \frac{\partial D^\alpha(f_0 * f)}{\partial x_j} = \frac{\partial((D^\alpha f_0) * f)}{\partial x_j} = \frac{\partial D^\alpha f_0}{\partial x_j} * f = (D^{\alpha+e_j} f_0) * f.$$

Inductively, we have $D^\alpha(f_0 * f) = (D^\alpha f_0) * f$ for all multi-index α .

Let $f \in [E^{(\gamma, U)}]$, $\xi \in E$ and $1 \leq j \leq n$. By Th. 4.2, $f_0 * f \in [E^{(\gamma, V)}]$ for some $V \in \mathcal{N}(E)$ and $\frac{\partial(f_0 * f)}{\partial x_j}(\xi) = (f_0 * f)\left(-\frac{\partial \xi}{\partial x_j}\right) = f[f_0 * (-\frac{\partial \xi}{\partial x_j})] = f(-f_0 * \frac{\partial \xi}{\partial x_j}) = f(-\frac{\partial f_0 * \xi}{\partial x_j}) = \frac{\partial f}{\partial x_j}(f_0 * \xi) = (f_0 * \frac{\partial f}{\partial x_j})(\xi)$, i.e., $\frac{\partial(f_0 * f)}{\partial x_j} = f_0 * \frac{\partial f}{\partial x_j}$.

If $D^\alpha(f_0 * f) = f_0 * D^\alpha f$, then $D^{\alpha+e_j}(f_0 * f) = \frac{\partial D^\alpha(f_0 * f)}{\partial x_j} = \frac{\partial f_0 * D^\alpha f}{\partial x_j} = f_0 * \frac{\partial D^\alpha f}{\partial x_j} = f_0 * D^{\alpha+e_j} f$ since $D^\alpha f \in [E^{(\gamma, W)}]$ for some $W \in \mathcal{N}(E)$ by Th. 2.1. \square

Recall that if $f \in E'$ for which $\text{supp } f$ is bounded in \mathbb{R}^n , then f must be a convolution multiplier [4, Th. 3.3.4]. Then we can develop the result of continuity of convolution [4, Th. 3.3.5].

First, we give an improvement of Th. 3.3.5 of [4] as follows.

Theorem 4.4 *If $\{f_k\} \subset E'$ such that $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \rightarrow f(\xi)$ at each $\xi \in E$ ($f \in E'$ by Th. 1.8) and there is a bounded $F \subset \mathbb{R}^n$ such that $\text{supp } f_k \subseteq F$, $\forall k \in \mathbb{N}$, then for every $g \in [E^{(\gamma, U)}]$ and bounded $B \subset E$,*

$$\lim_k (f_k * g)(\xi) = (f * g)(\xi) \text{ uniformly for } \xi \in B.$$

Proof. By Th. 1.9, $\text{supp } f \subseteq F$ and so f is also a convolution multiplier on E . By Th. 1.2, for every bounded $B \subset E$, $\lim_k f_k(\xi) = f(\xi)$ uniformly for $\xi \in B$. Then $f_k * \xi \rightarrow f * \xi$, $\forall \xi \in E$ (see the proof of Th 3.3.5 of [4]).

Define $T : E \rightarrow E$ and $T_k : E \rightarrow E$ by $T(\xi) = f * \xi$ and $T_k(\xi) = f_k * \xi$, $\forall \xi \in E$, $k \in \mathbb{N}$. By Lemma 4.1, T and all T_k are continuous and linear.

Since \mathcal{D} is an (LF) space and \mathcal{S} is a locally convex Fréchet space, both \mathcal{D} and \mathcal{S} are barrelled [3, p.136, 222]. Moreover, $T_k(\xi) = f_k * \xi \rightarrow f * \xi = T(\xi)$ at each $\xi \in E$, i.e., $\{T_k(\xi) : k \in \mathbb{N}\}$ is bounded at each $\xi \in E$. By Th. 9.3.4 of [3], both $\{T_k : k \in \mathbb{N}\}$ and $\{T_k : k \in \mathbb{N}\} \cup \{T\}$ are equicontinuous on E .

Let $g \in [E^{(\gamma, U)}]$. Pick a $V \in \mathcal{N}(E)$ for which $T(V) \subset U$ and $T_k(V) \subset U$, $\forall k \in \mathbb{N}$, i.e., $f * \eta, f_k * \eta \in U$, $\forall \eta \in V$, $k \in \mathbb{N}$. Then $f * g, f_k * g \in [E^{(\gamma, V)}]$ for all k (see the proof of Th. 4.2).

Suppose that $g \in E^{(\gamma, U)}$. Then $f * g, f_k * g \in E^{(\gamma, V)}$ and

$$(f_k * g)(\xi) = g(f_k * \xi) \rightarrow g(f * \xi) = (f * g)(\xi), \forall \xi \in E.$$

By Th. 1.2, for every bounded $B \subset E$, $\lim_k (f_k * g)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$.

Now let $g = \sum_{\nu=1}^m a_\nu g_\nu$ where $a_\nu \in \mathbb{C}$ and $g_\nu \in E^{(\gamma, U)}$, $\nu = 1, 2, \dots, m$. Then for every bounded $B \subset E$ we have that

$$\begin{aligned} \lim_k (f_k * g)(\xi) &= \lim_k g(f_k * \xi) = \lim_k \left(\sum_{\nu=1}^m a_\nu g_\nu \right) (f_k * \xi) = \lim_k \sum_{\nu=1}^m a_\nu g_\nu (f_k * \xi) \\ &= \lim_k \sum_{\nu=1}^m a_\nu (f_k * g_\nu)(\xi) = \sum_{\nu=1}^m a_\nu \lim_k (f_k * g_\nu)(\xi) = \sum_{\nu=1}^m a_\nu (f * g_\nu)(\xi) = (f * g)(\xi) \end{aligned}$$

uniformly for $\xi \in B$. \square

We also give some simple facts before our main result Th. 4.6.

Theorem 4.5 *Let $f_0 \in E'$ be a convolution multiplier, $U \in \mathcal{N}(E)$ and $\gamma \in C(0)$. There is a $V \in \mathcal{N}(E)$ for which $f_0 * \cdot : [E^{(\gamma, U)}] \rightarrow [E^{(\gamma, V)}]$ is a linear operator such that $f_0 * f \in E^{(\gamma, V)}$ for each $f \in E^{(\gamma, U)}$. Moreover, if $f_k \xrightarrow{w*} f$ in $E^{(\gamma, U)}$, i.e., $f, f_k \in E^{(\gamma, U)}$ and $f_k(\xi) \rightarrow f(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_k (f_0 * f_k)(\xi) = (f_0 * f)(\xi)$ uniformly for $\xi \in B$.*

Proof. By Th. 4.2, there is a $V \in \mathcal{N}(E)$ such that $f_0 * f \in E^{(\gamma, V)}$ for $f \in E^{(\gamma, U)}$ and so $f_0 * f \in [E^{(\gamma, V)}]$ whenever $f \in [E^{(\gamma, U)}]$.

Let $f, g \in [E^{(\gamma, U)}]$ and $t \in \mathbb{C}$. Then $[f_0 * (f + tg)](\xi) = (f + tg)(f_0 * \xi) = f(f_0 * \xi) + tg(f_0 * \xi) = [(f_0 * f) + t(f_0 * g)](\xi)$, $\forall \xi \in E$, i.e., $f_0 * (f + tg) = f_0 * f + t(f_0 * g)$ and so $f_0 * \cdot$ is a linear operator.

Let $f_k \xrightarrow{w*} f$ in $E^{(\gamma, U)}$. Then $(f_0 * f_k)(\xi) = f_k(f_0 * \xi) \rightarrow f(f_0 * \xi) = (f_0 * f)(\xi)$, $\forall \xi \in E$. By Th. 1.2, for every bounded $B \subset E$, $\lim_k (f_0 * f_k)(\xi) = (f_0 * f)(\xi)$ uniformly for $\xi \in B$. \square

Corollary 4.1 Suppose that $f_k \xrightarrow{w*} f$ in E' where each $\text{supp } f_k \subset \{x \in \mathbb{R}^n : |x| \leq a\}$ for some $a > 0$ and $g_k \xrightarrow{w*} g$ in $E^{(\gamma, U)}$. If $\xi_k \rightarrow \xi$ in E , then

$$(f_k * h)(\xi_k) \rightarrow (f * h)(\xi), \quad \forall h \in [E^{(\gamma, U)}],$$

$$(f_0 * g_k)(\xi_k) \rightarrow (f_0 * g)(\xi), \quad \forall \text{ convolution multiplier } f_0 \in E',$$

Proof. Since $\xi_m \rightarrow \xi$, $\{\xi_m : m \in \mathbb{N}\}$ is bounded in E .

Let $h \in [E^{(\gamma, U)}]$. By Th. 4.4, $\lim_k (f_k * h)(\xi_m) = (f * h)(\xi_m)$ uniformly for $m \in \mathbb{N}$. But $\lim_m (f_k * h)(\xi_m) = (f_k * h)(\xi)$ for each $k \in \mathbb{N}$ and so $\lim_k (f_k * h)(\xi_k) = \lim_{k, m \rightarrow +\infty} (f_k * h)(\xi_m) = \lim_k \lim_m (f_k * h)(\xi_m) = \lim_k (f_k * h)(\xi) = (f * h)(\xi)$.

Similarly, it follows from Th. 4.5 that for every convolution multiplier $f_0 \in E'$ we have $(f_0 * g_k)(\xi_k) \rightarrow (f_0 * g)(\xi)$. \square

Corollary 4.2 If $f_0 \in E'$ is a convolution multiplier and $P(D) = \sum_{|\alpha| \leq p} a_\alpha D^\alpha$, then

$$P(D)(f_0 * f) = f_0 * [P(D)f], \quad \forall f \in [E^{(\gamma, U)}].$$

Proof. Let $f \in [E^{(\gamma, U)}]$. By Th. 2.1, there is a $V \in \mathcal{N}(E)$ such that $D^\alpha f \in [E^{(\gamma, V)}]$, $\forall |\alpha| \leq p$. By Th. 4.2, there is a $W \in \mathcal{N}(E)$ such that $f_0 * D^\alpha f \in [E^{(\gamma, W)}]$, $\forall |\alpha| \leq p$. By Th. 4.3, Lemma 4.2 and Th. 4.5,

$$\begin{aligned} P(D)(f_0 * f) &= \sum_{|\alpha| \leq p} a_\alpha D^\alpha (f_0 * f) = \sum_{|\alpha| \leq p} a_\alpha (f_0 * D^\alpha f) = \sum_{|\alpha| \leq p} f_0 * (a_\alpha D^\alpha f) \\ &= f_0 * \left(\sum_{|\alpha| \leq p} a_\alpha D^\alpha f \right) = f_0 * [P(D)f]. \quad \square \end{aligned}$$

We now have a strong continuity result for convolution as follows.

Theorem 4.6 Let $\{f_k\} \subset E'$ be a sequence of usual distributions such that $f_k \xrightarrow{w*} f$, i.e., $f_k(\xi) \rightarrow f(\xi)$ at each $\xi \in E$ ($f \in E'$ by Th. 1.8) and there is a bounded $F \subset \mathbb{R}^n$ such that $\text{supp } f_k \subseteq F$, $\forall k \in \mathbb{N}$. If $g_k \xrightarrow{w*} g$ in $E^{(\gamma, U)}$, i.e., $g, g_k \in E^{(\gamma, U)}$ for all k and $g_k(\xi) \rightarrow g(\xi)$ at each $\xi \in E$, then for every bounded $B \subset E$, $\lim_{k, m \rightarrow +\infty} (f_k * g_m)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$ and, in particular, $\lim_k (f_k * g_k)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$, and $(f_k * g_k)(\xi_k) \rightarrow (f * g)(\xi)$ whenever $\xi_k \rightarrow \xi$ in E .

Proof. As in the proof of Th. 4.4, it follows from $f_k \xrightarrow{w*} f$ in E' and $g, g_k \in E^{(\gamma, U)}$ that there is a $V \in \mathcal{N}(E)$ such that $f * g, f_m * g_k \in E^{(\gamma, V)}$ for all $k, m \in \mathbb{N}$.

Let $\xi \in E$. As was noticed in the proof of Th. 4.4, $f_m * \xi \rightarrow f * \xi$ in E and so $\lim_m (f_m * g_k)(\xi) = \lim_m g_k(f_m * \xi) = g_k(f * \xi)$, $\forall k \in \mathbb{N}$. But $\{f_m * \xi\}$ is bounded in E and, by Th. 1.2, $\lim_k (f_m * g_k)(\xi) = \lim_k g_k(f_m * \xi) = g(f_m * \xi) = (f_m * g)(\xi)$ uniformly for $m \in \mathbb{N}$. Then $\lim_{k, m \rightarrow +\infty} (f_m * g_k)(\xi) = \lim_m \lim_k (f_m * g_k)(\xi) = \lim_m (f_m * g)(\xi) = \lim_m g(f_m * \xi) = g(f * \xi) = (f * g)(\xi)$, $\forall \xi \in E$.

Let B be a bounded subset of E . If $\lim_{k, m \rightarrow +\infty} (f_m * g_k)(\xi) = (f * g)(\xi)$ is not uniformly for $\xi \in B$, then there exist $\varepsilon > 0$, $\{\xi_\nu\} \subset B$ and integer sequences $k_1 < k_2 < \dots$ and $m_1 < m_2 < \dots$ such that

$$(*) \quad |(f_{m_\nu} * g_{k_\nu})(\xi_\nu) - (f * g)(\xi_\nu)| \geq \varepsilon, \quad \nu = 1, 2, 3, \dots$$

Since $f * g, f_{m_\nu} * g_{k_\nu} \in E^{(\gamma, V)}$ for all $\nu \in \mathbb{N}$ and

$$\lim_\nu (f_{m_\nu} * g_{k_\nu})(\xi) = \lim_{k, m \rightarrow +\infty} (f_m * g_k)(\xi) = (f * g)(\xi), \quad \forall \xi \in E,$$

it follows from Th. 1.2 or Th. 1.7 that $\lim_{\nu}(f_{m_{\nu}} * g_{k_{\nu}})(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$ and so there is a $\nu_0 \in \mathbb{N}$ such that

$$|(f_{m_{\nu}} * g_{k_{\nu}})(\xi_{\nu}) - (f * g)(\xi_{\nu})| < \varepsilon, \quad \forall \nu > \nu_0.$$

This contradicts (*) and so $\lim_{k,m \rightarrow +\infty}(f_m * g_k)(\xi) = (f * g)(\xi)$ uniformly for $\xi \in B$. \square

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